Mathematical Methods in Origami

University of Tokyo, Komaba
Day 3 Slides, Dec. 18, 2015
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What does NP-hard mean?

- Some problems can be solved very efficiently with an algorithm.
(I.e., if the number of inputs is n, then the running time of the algorithm will be a polynomial in terms of n, like n^2 or $n^{10}+n$ or whatev.)

⟶ class P (for polynomial)

- But there are many problems, like the Traveling Salesperson Problem (TSP), for which no one has ever been able to find a polynomial running-time algorithm.
(I.e., the running time is exponential or worse.)

- Theoretical computer scientists have found ways to prove that more difficult problems (like the TSP) are **equivalent**, in that if you could find a polynomial-time algorithm for one of them, you could translate it to ALL the other hard problems.

⟶ class NP (for non-deterministic polynomial)

- Any problem that is shown to be equivalent to a problem in NP is called **NP-hard**.

What does this have to do with origami?

- **The flat foldability problem:** Given a bunch of lines drawn on a piece of paper, can they be the crease pattern for a flat origami model? (I.e., can they fold flat?)

- In 1996, Marshall Bern & Barry Hayes proved that the flat foldability problem in NP-hard!

- Actually, they proved two things:

  Theorem 1: Determining if a crease pattern is flat-foldable is NP-hard.
  Proof is elegant and very cool from an origami perspective.

  Theorem 2: Given a crease pattern with mountains and valleys specified, it's STILL NP-hard to tell if its flat-foldable.
  Proof is whaaa? hard to understand.

How'd they prove flat-foldability is NP-hard?

- They show that solving Not-All-Equal 3-Satisfiability (which is NP-hard) can be reduced to flat-foldability.

- NAE 3-SAT: Given boolean variables $x_1, x_2, x_3, \ldots, x_n$
  and a bunch of clause functions of three variables

$$C(a, b, c) = \begin{cases} T & \text{if not all of } a, b, c \text{ have the same value} \\ F & \text{otherwise} \end{cases}$$

  can you find values for the variables that make all the clauses true?

- Example: With variables x_1, x_2, x_3, x_4 and clauses

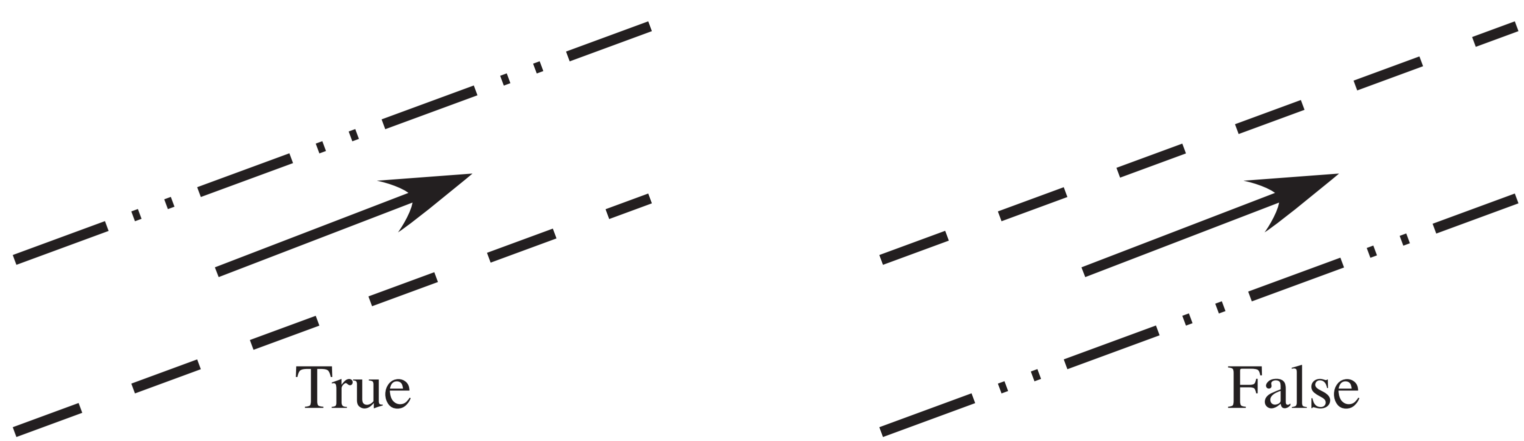
$$C(x_1, x_2, x_3), C(x_2, x_3, x_4), C(\tilde{x}_1, x_3, x_4)$$

  can NAE 3SAT work?

How'd they prove flat-foldability is NP-hard?

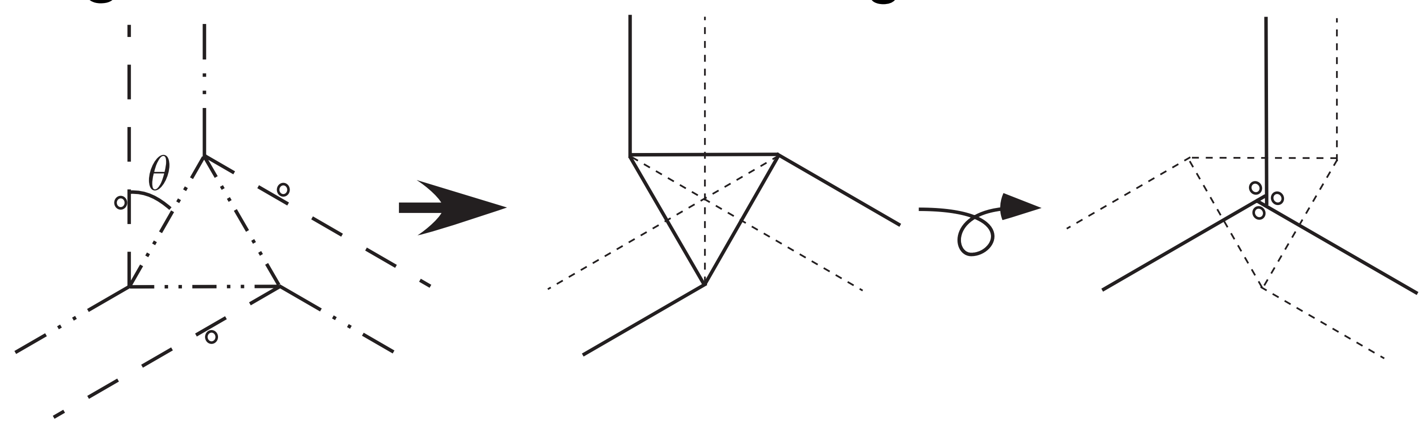
- Goal: Given any NAE 3-SAT problem, we want to create an origami crease pattern with the property that the crease pattern will fold flat if and only if the NAE 3-SAT problem works.

- We will use **pleats** in the paper to represent "wires" that can be either True or False.
  Each wire will be given an initial **direction**, and whether the pleat is MV or VM will determine its truth value.

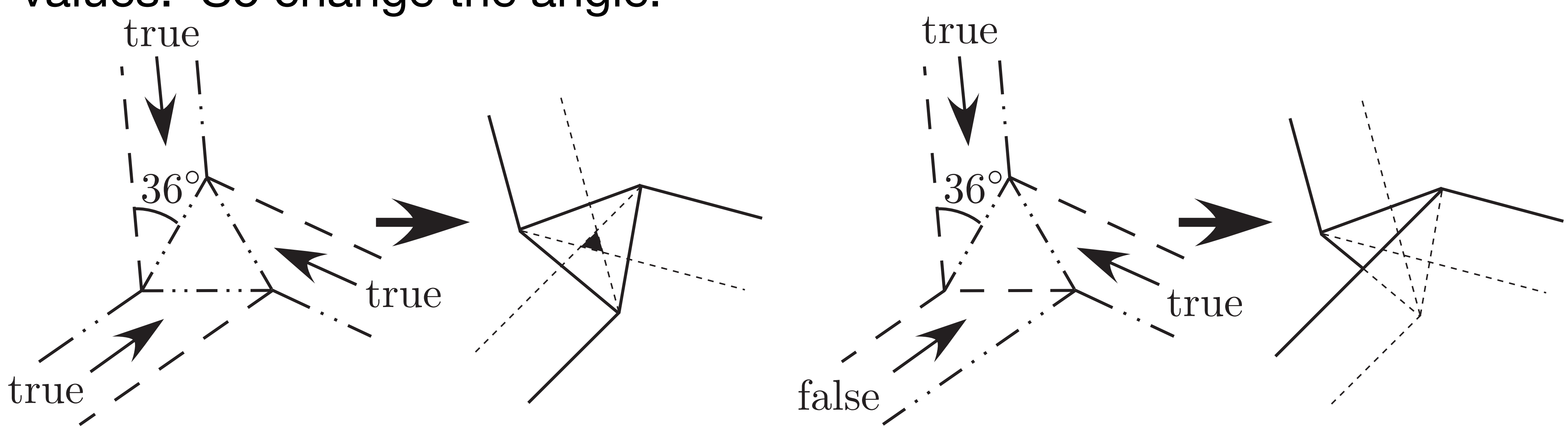


How'd they prove flat-foldability is NP-hard?

- **Clause gadgets:** Remember how triangle twists work?



- A normal triangle twist will admit any collection of entering wire truth values. So change the angle!



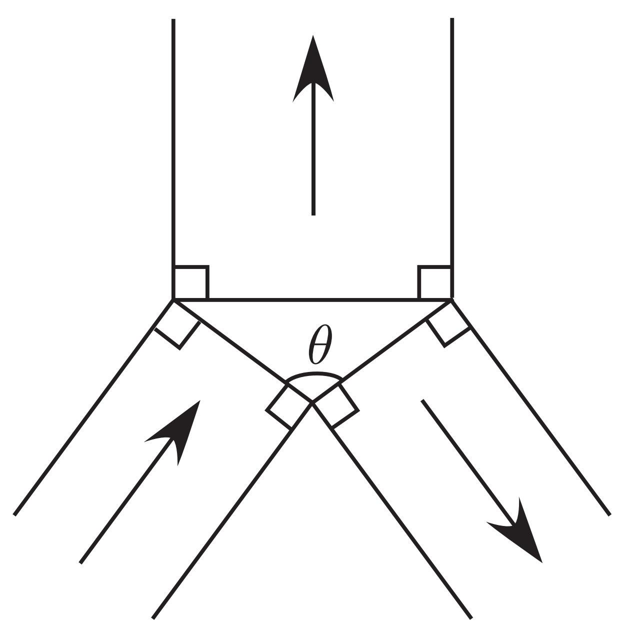
This mimics exactly how the $C(x_i, x_j, x_k)$ clauses work for NAE 3-SAT. (I.e., it folds flat if and only if the incoming wires are not all the same truth value!)

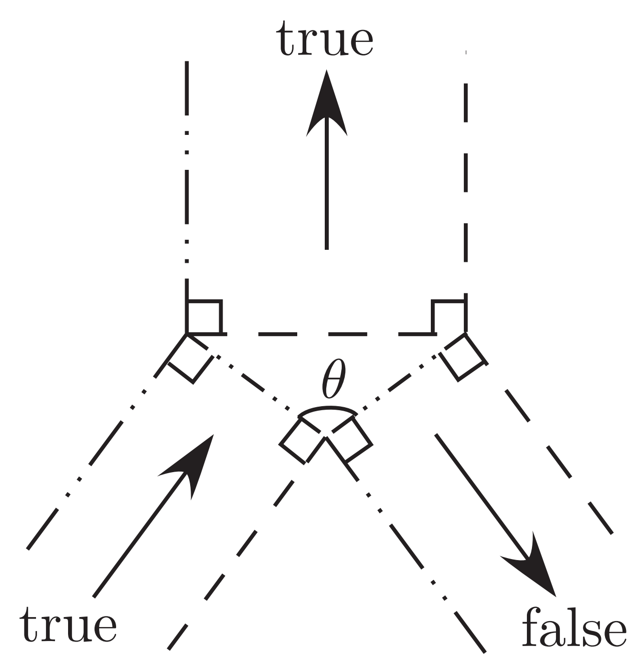
How'd they prove flat-foldability is NP-hard?

- What else do we need?

- Gadgets to deflect wires and/or change their truth value.

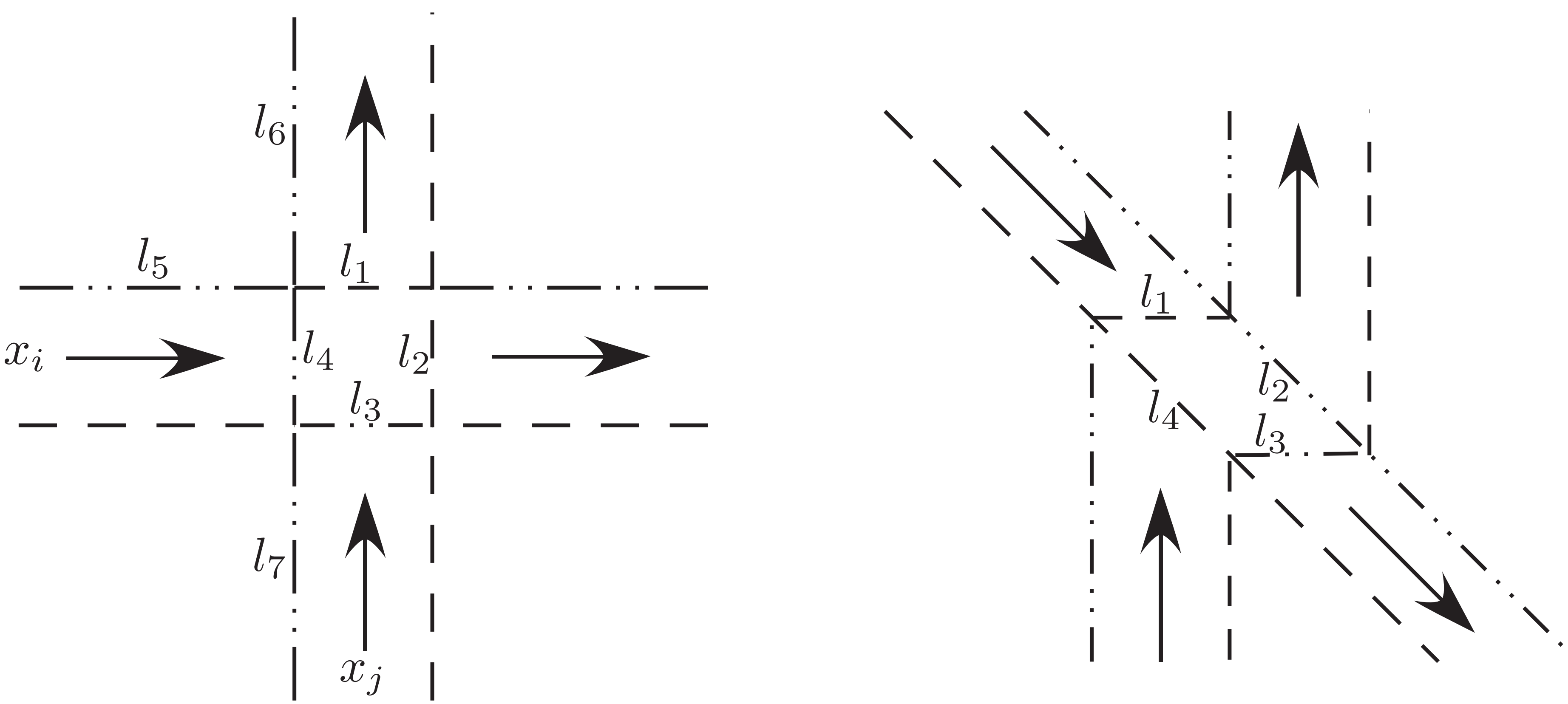
Reflector gadget:





How'd they prove flat-foldability is NP-hard?

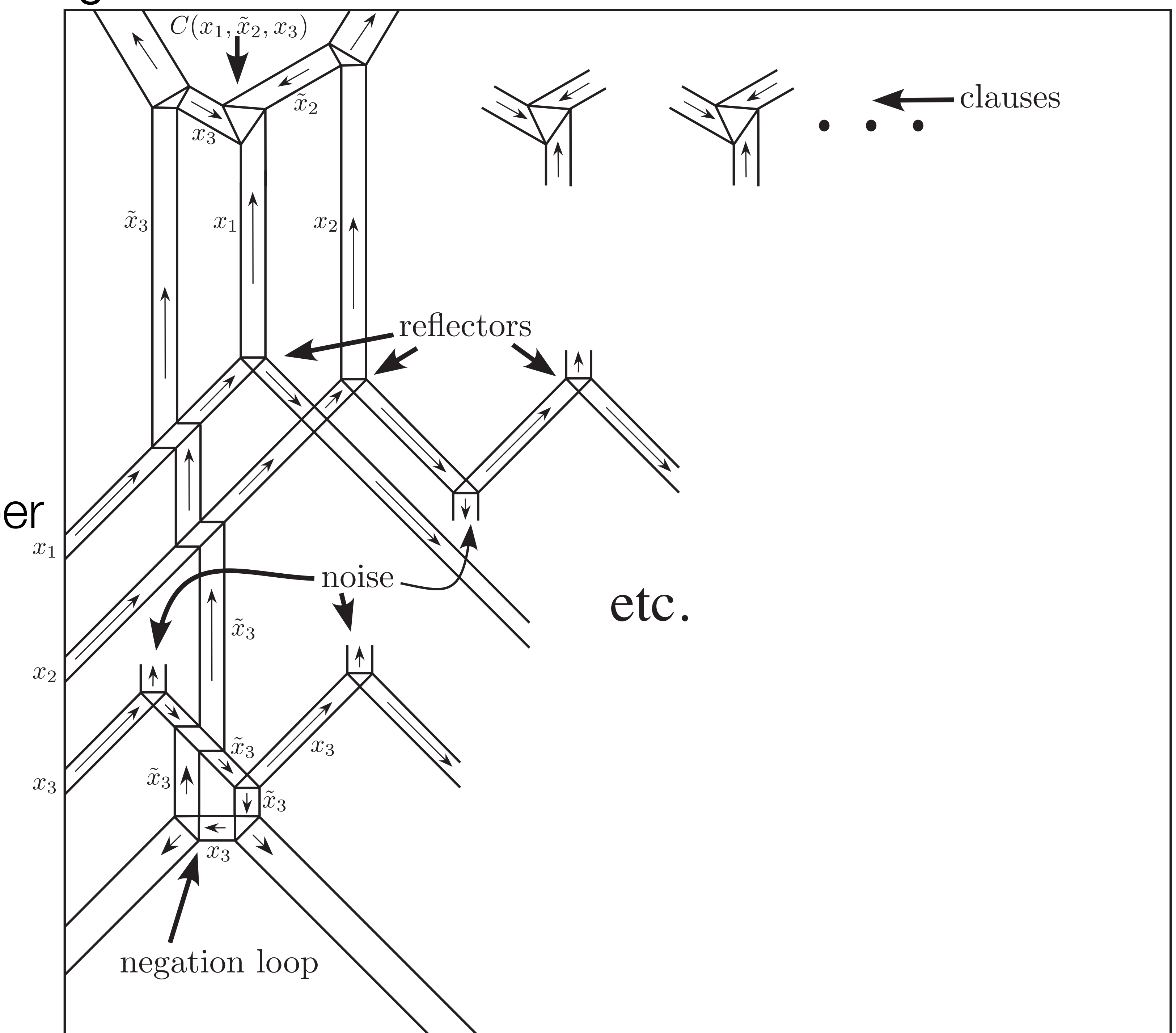
- What else do we need?
- To be able to handle cases where two wires intersect. We don't want them to interfere with each other!



How’d they prove flat-foldability is NP-hard?

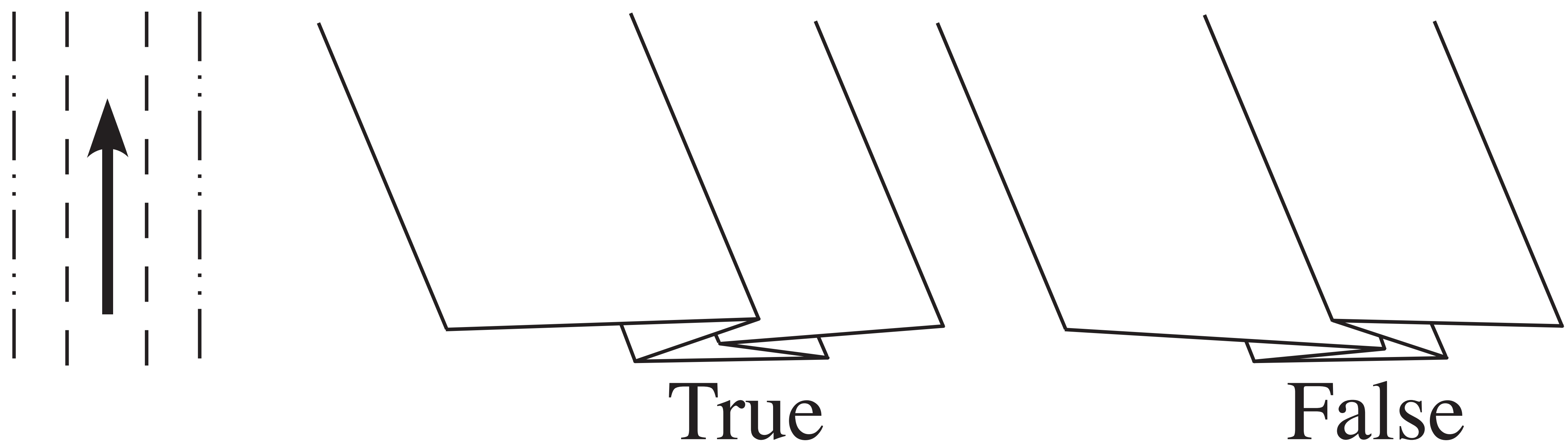
- Putting it all together:

We start with all our wires on the left side of the paper going up at the same angle.



What about if we are given the MV assignment?

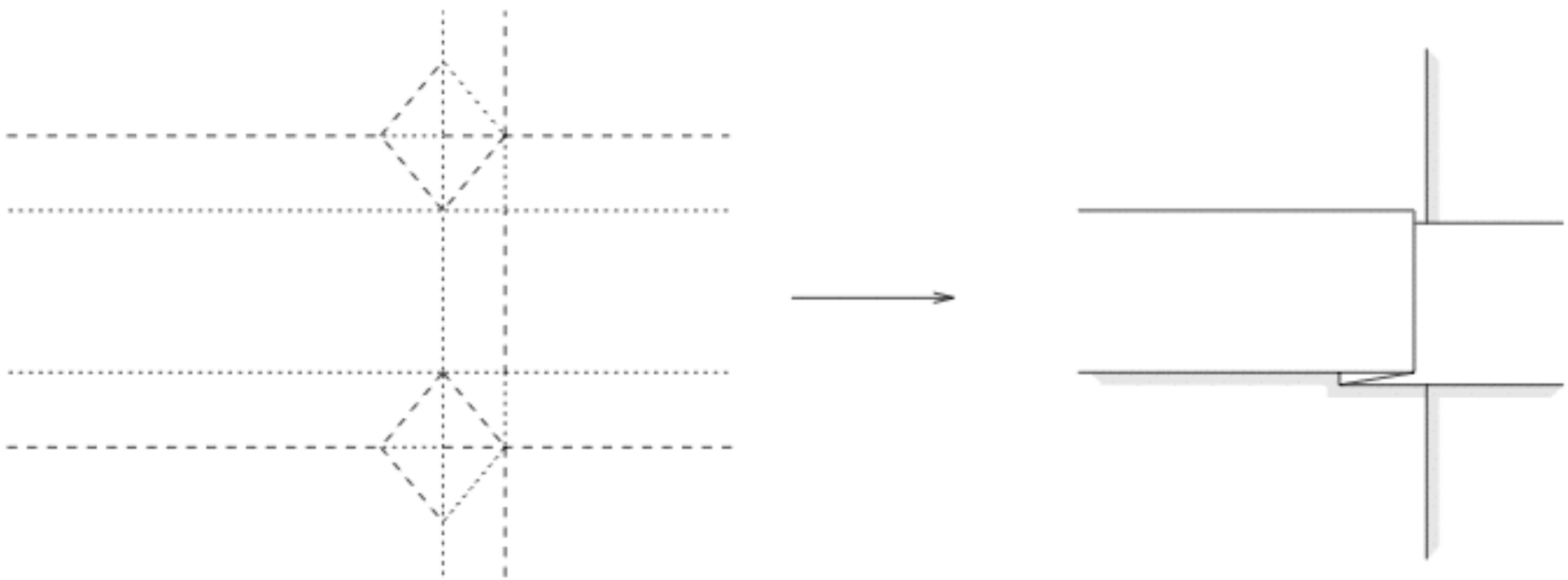
- The previous proof uses mountains and valleys to determine the truth values of the wires. If the Ms and Vs are given to us at the outset, then we have to make wires whose truth value is **not** determined by Ms and Vs, but by the **layers** of the paper when folded flat.



All that's needed are Not-All-Equal clause gadgets and a "signal splitter" or reflector gadgets to work with these wires.

What about if we are given the MV assignment?

- Bern & Hayes' gadgets for this proof are **complicated**!
- They require that we first produce lots of "flaps":



What about if we are given the MV assignment?

- Bern & Hayes‘ gadgets for this proof are **complicated**!
- They require that we first produce lots of “flaps.”

Actually, a group of us discovered a flaw in Bern & Hayes’ proof.

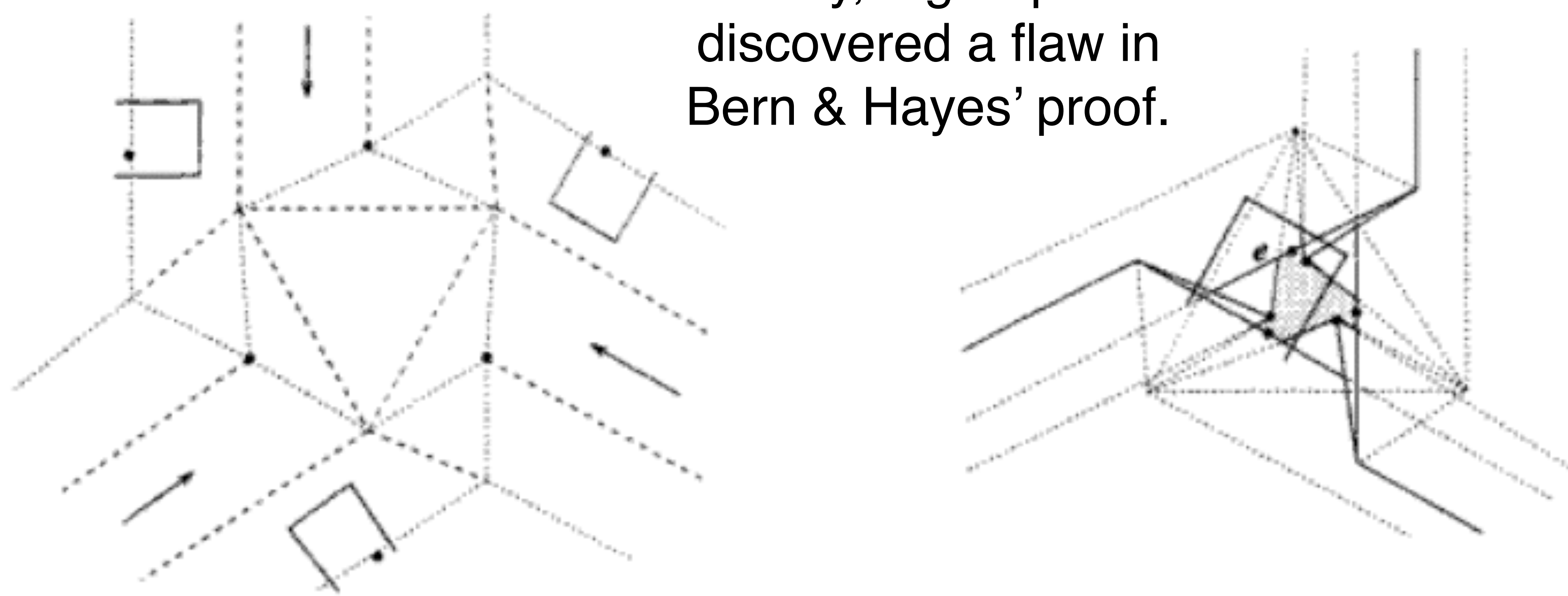
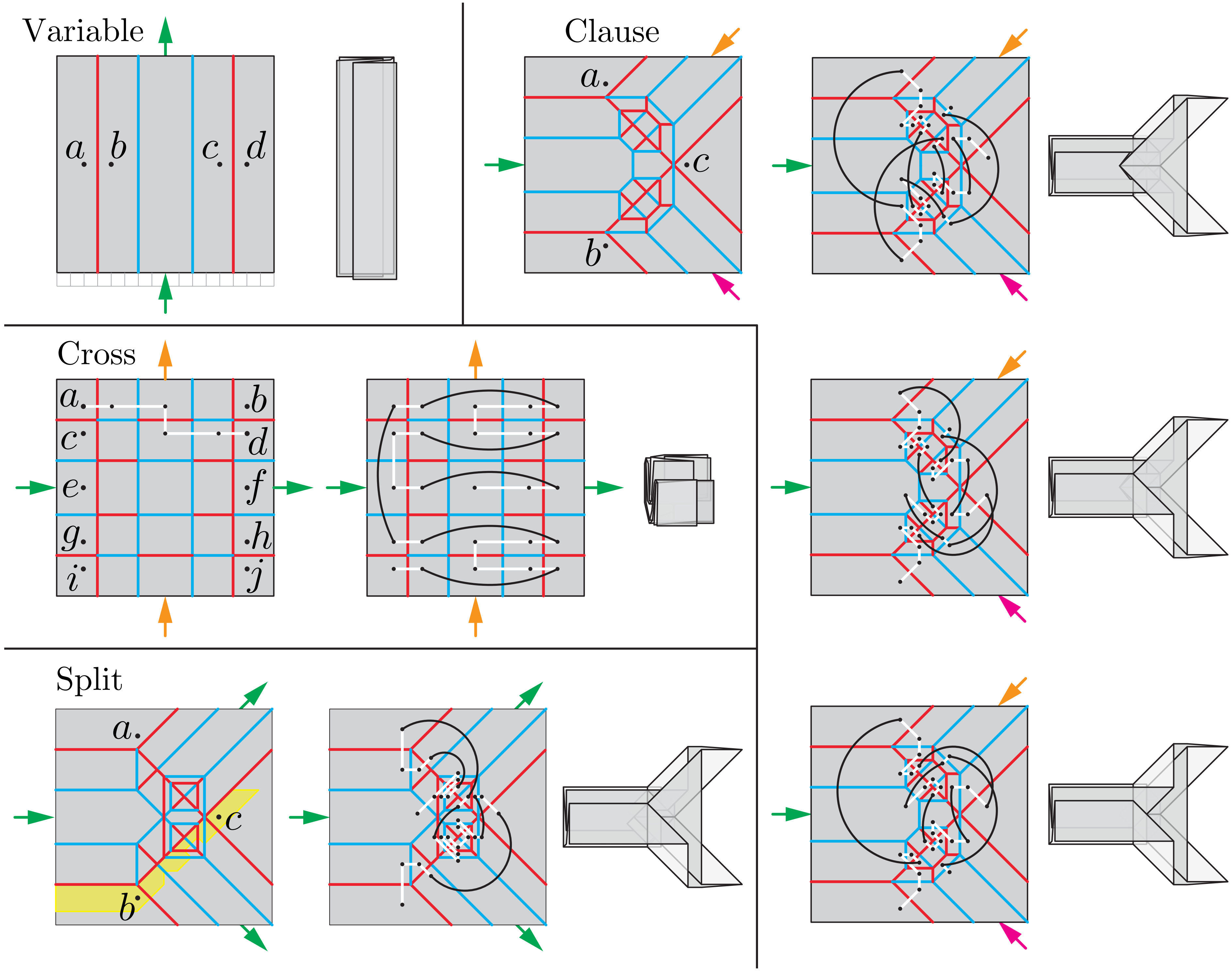


Figure 12: (a) Clause gadget for ASSIGNED FLAT FOLDABILITY. (b) Folded (only one tab shown).

What about if we are given the MV assignment?

- A group of us (me, Tachi, Akitaya, Cheung, Demaine, Horiyama, Ku, and Uehara) found this flaw in the assigned case of Bern & Hayes' proof and proved a stronger result: That this problem is still NP-Hard if we restrict ourselves to **box pleating** (all creases are either horizontal, vertical, or at 45° angles).



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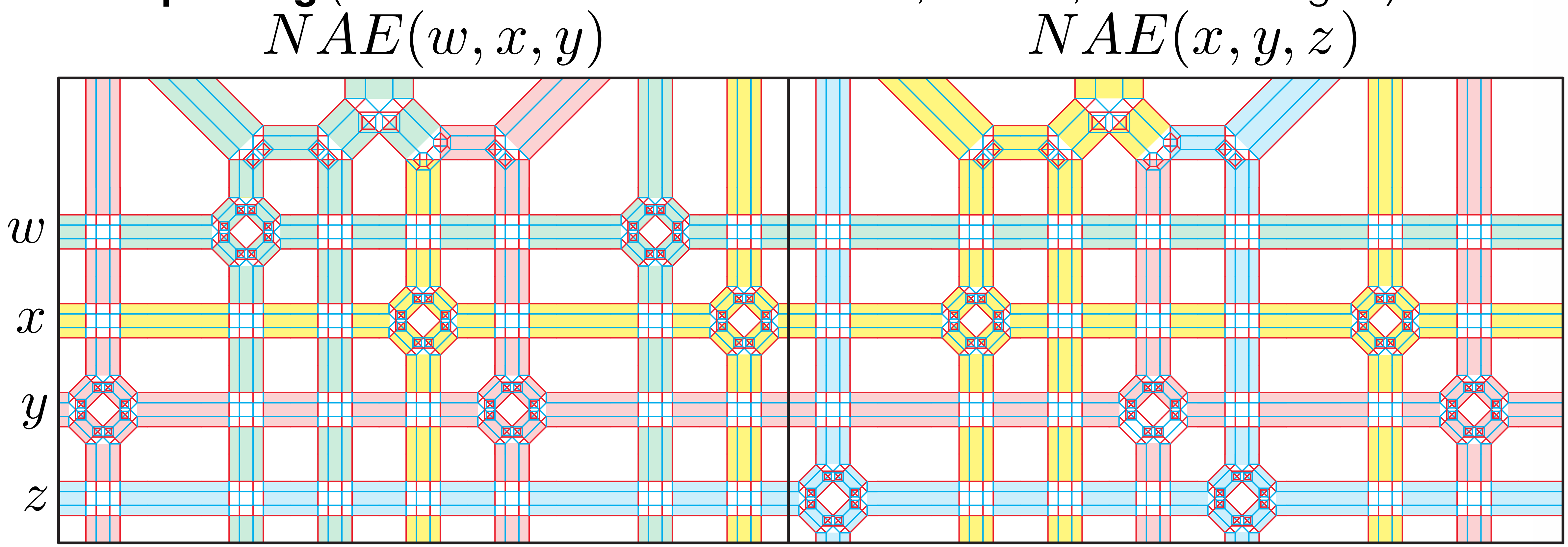
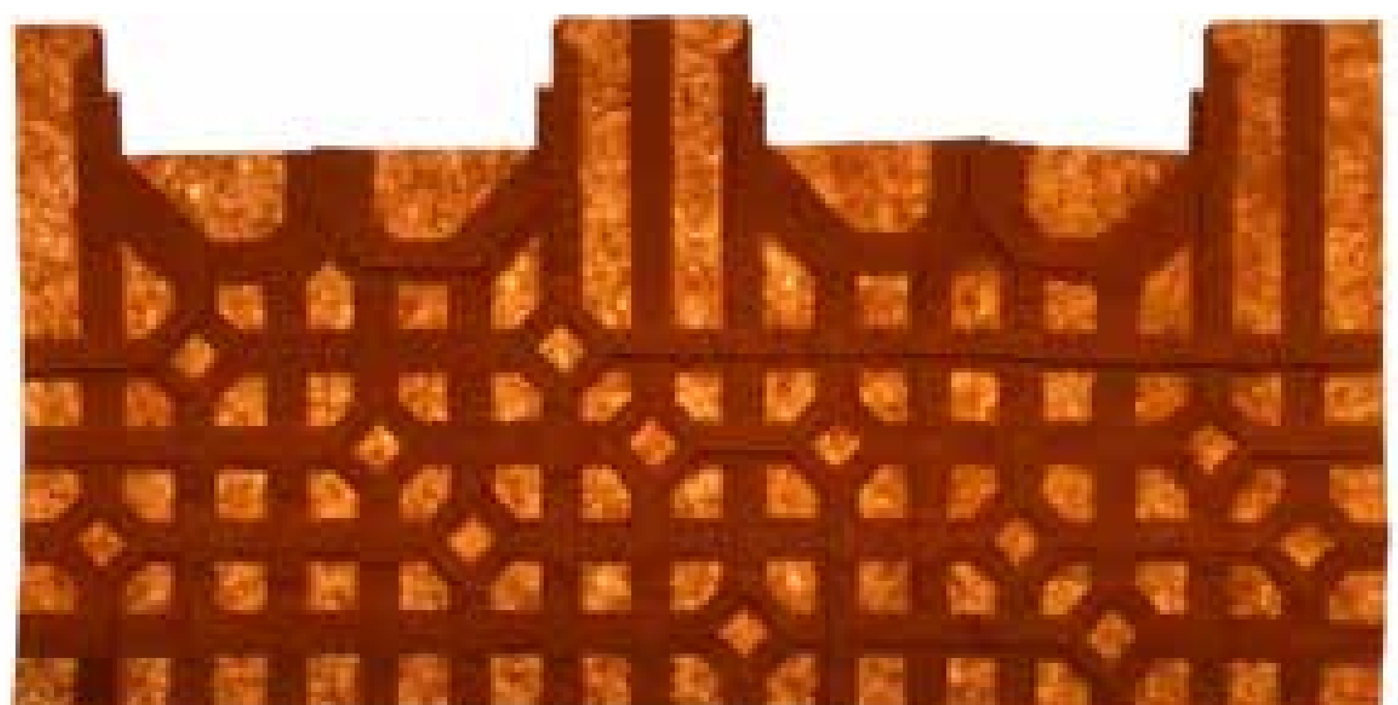
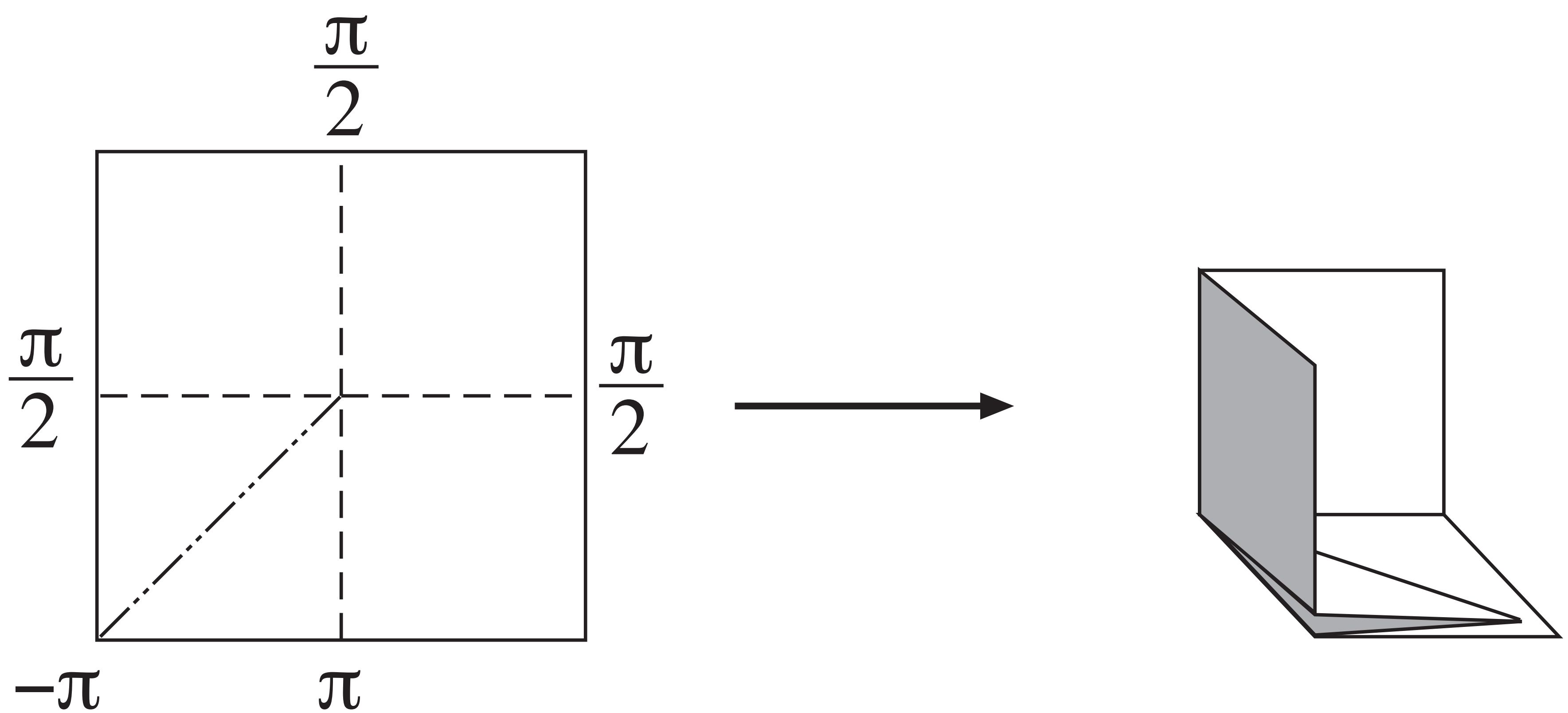




3D Rigid Foldings

- Are there Maekawa or Kawasaki-like Theorems for a rigidly-foldable vertex?



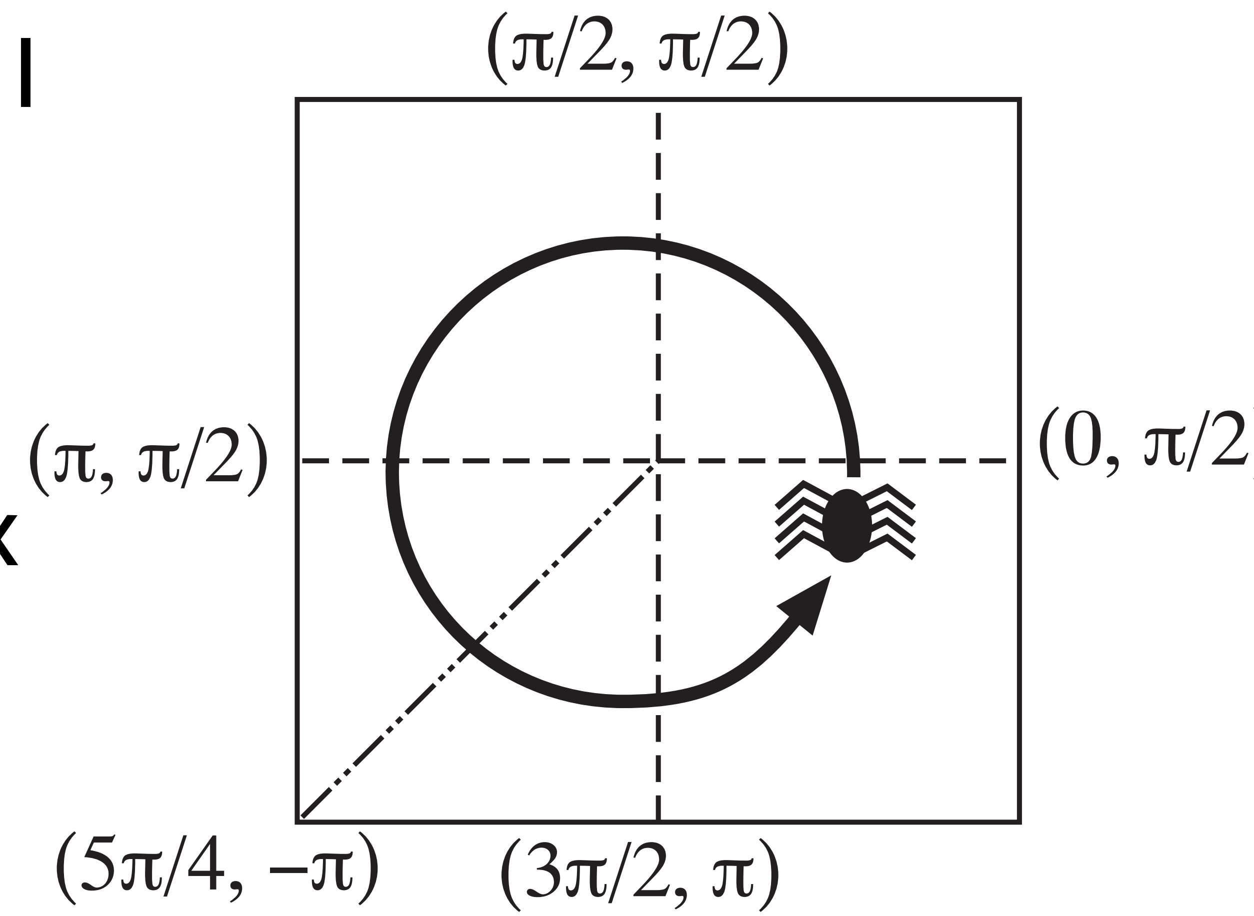
Note: The folding angles need to be specified at each crease, not just M or V.

3D Rigid Foldings

- **Theorem:** Let $R(L_i)$ = the counterclockwise rotation by folding angle p_i matrix about the axis given by unfolded line L_i. Then if the vertex rigidly folds, we will have

$$R(L_1)R(L_2)\dots R(L_n) = I$$

Proof: Follow the spider. The rotations it makes as it crawls around the vertex must have product equal to the identity matrix. BUT these rotations are not the $R(L_i)$ matrices!



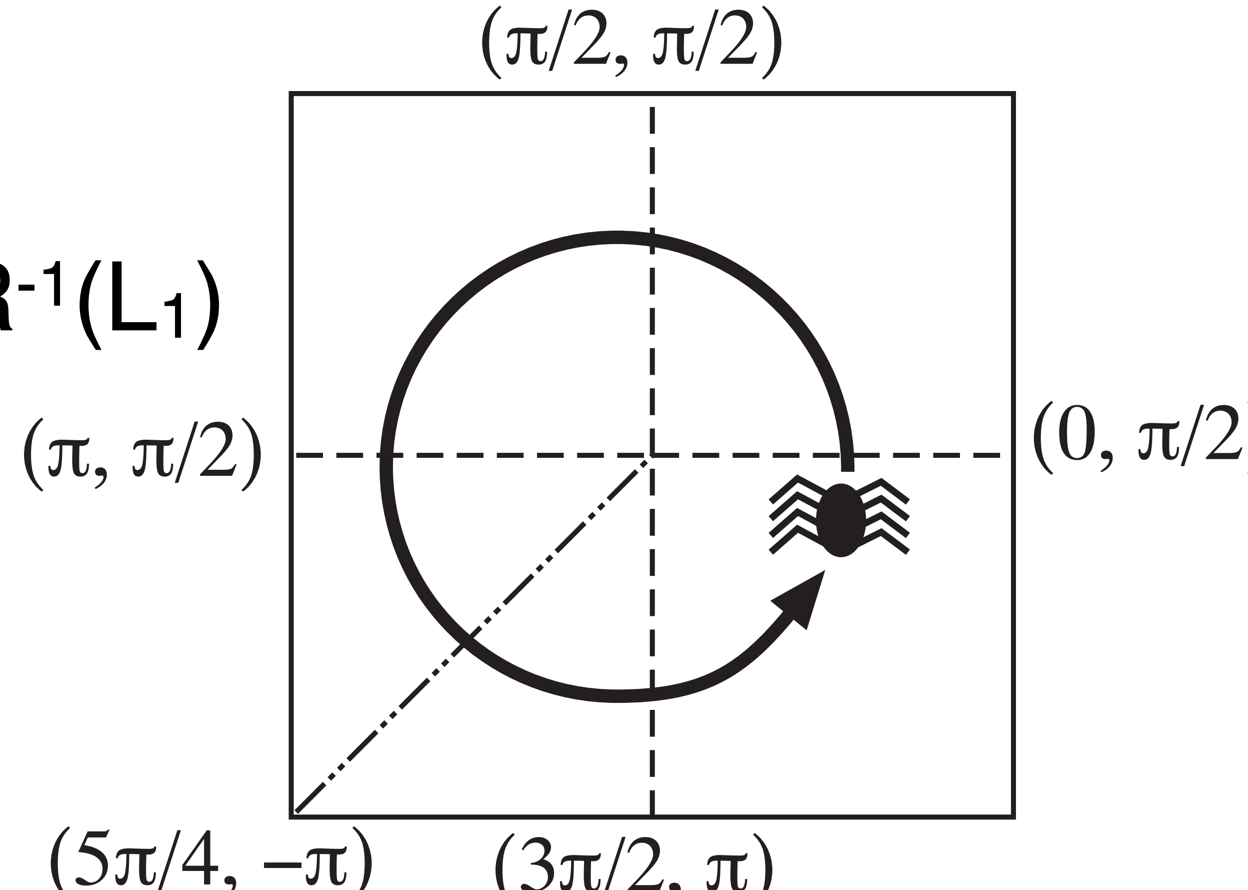
3D Rigid Foldings

The spider is crawling on the folded paper.
Let M_i = the rotation the spider does as it crawls around crease L_i.
Then, assume the face between L_1 and L_n is fixed…

$M_1 = R(L_1)$
$M_2 = R(L_1)\, R(L_2)\, R^{-1}(L_1)$
$M_3 = R(L_1)\, R(L_2)\, R(L_3)\, R^{-1}(L_2)\, R^{-1}(L_1)$

and

M_i = (redo previous L_is) $R(L_i)$ (undo previous L_is)

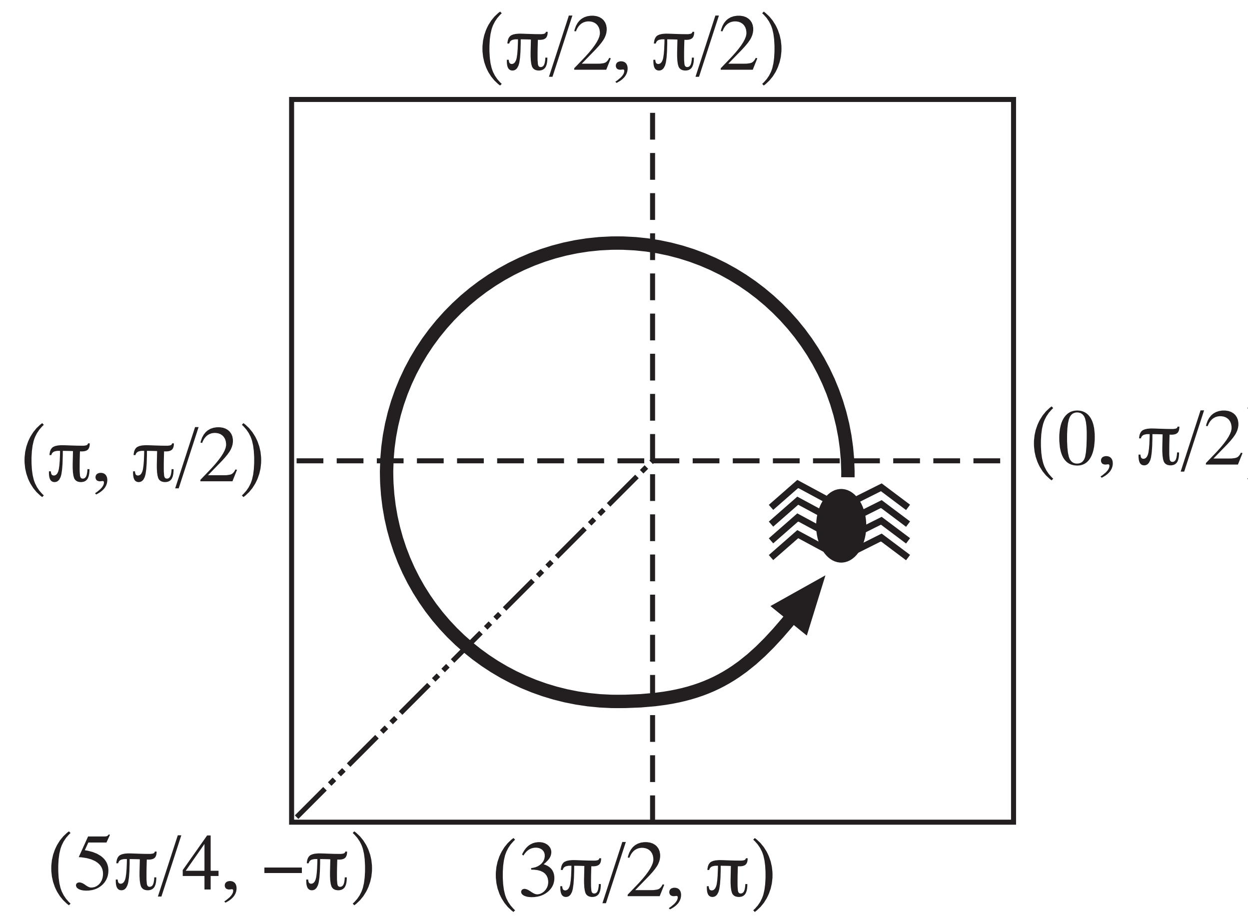


Then we'll have $M_1\, M_2\, M_3 \ldots M_n = I$.

3D Rigid Foldings

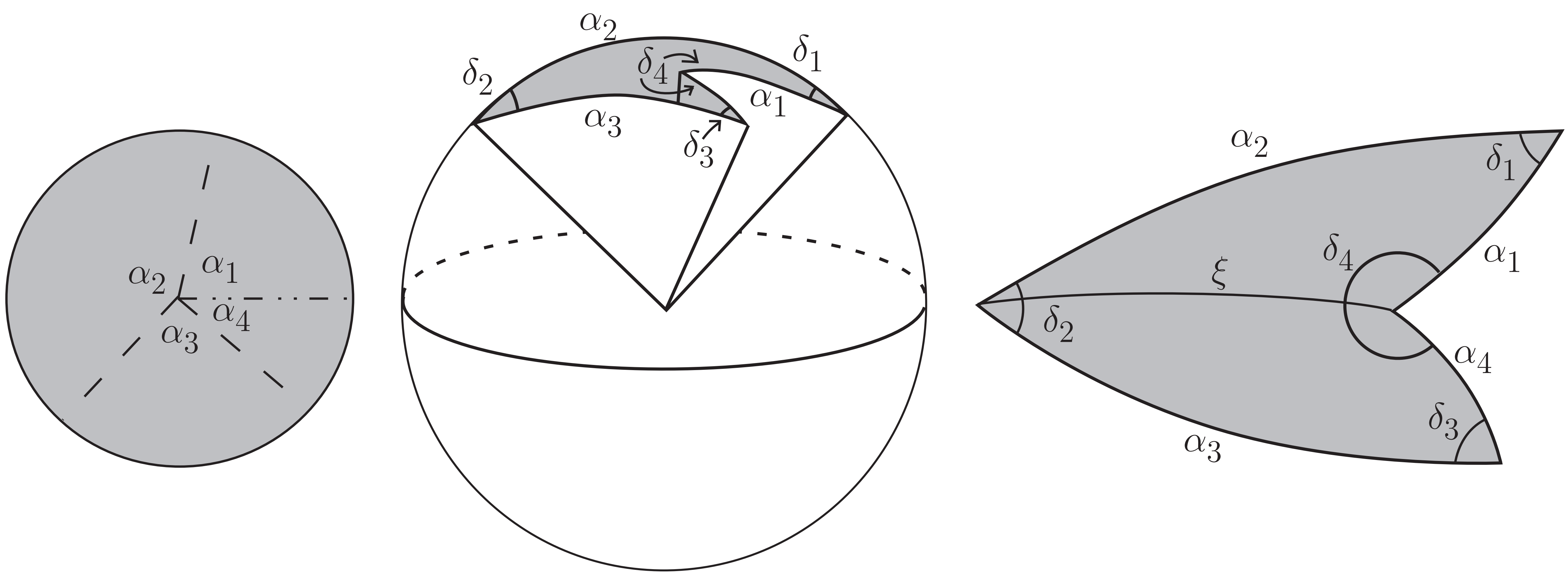
But if we actually compute M_1 M_2 M_3 … M_n lots of stuff cancels! It leaves us with

$$R(L_1)R(L_2)\ldots R(L_n) = I.$$



Special degree 4 case!

Oooo! Imagine the degree 4 rigidly-folded vertex projected onto a sphere.

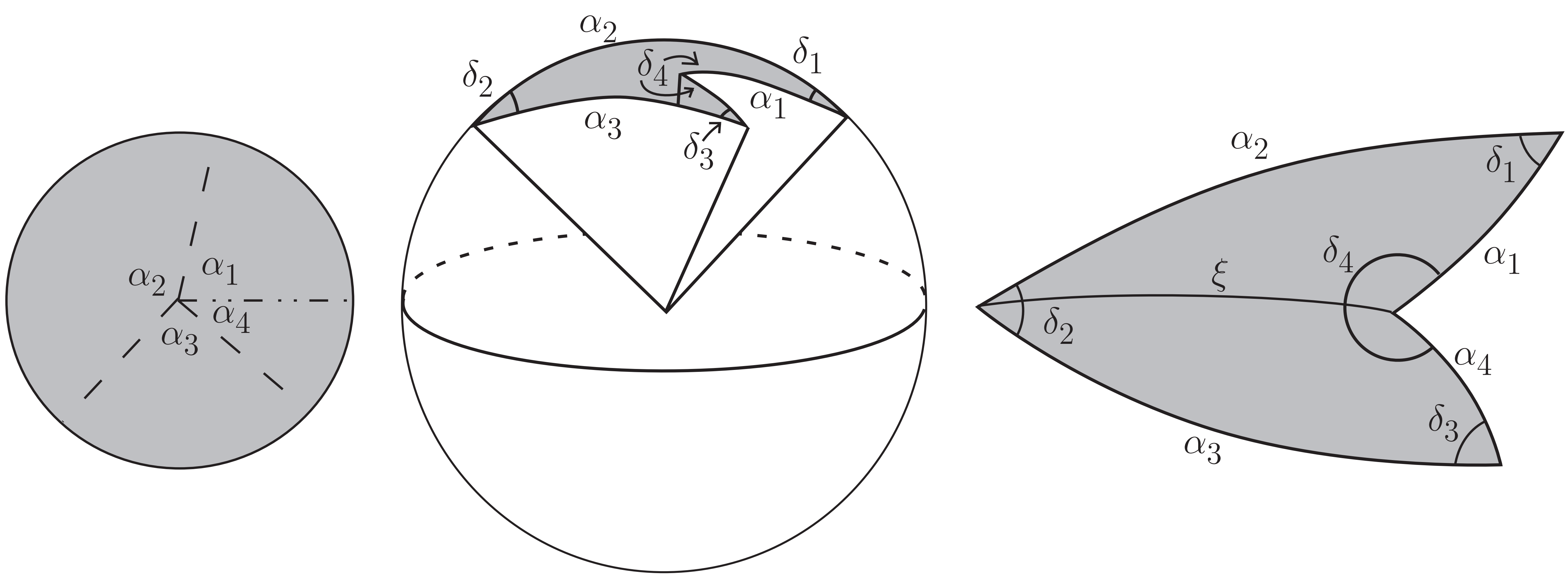


We get a spherical polygon with dihedral angles δ_i .
Divide this polygon into two spherical triangles.

Special degree 4 case!

Then by the spherical law of cosines:

$$\cos\xi = \cos\alpha_1\cos\alpha_2 + \sin\alpha_1\sin\alpha_2\cos\delta_1$$
$$\cos\xi = \cos\alpha_3\cos\alpha_4 + \sin\alpha_3\sin\alpha_4\cos\delta_3$$



Now, if this vertex folds flat, then

$$\alpha_3 = \pi - \alpha_1 \qquad\qquad \alpha_4 = \pi - \alpha_2$$

Special degree 4 case!

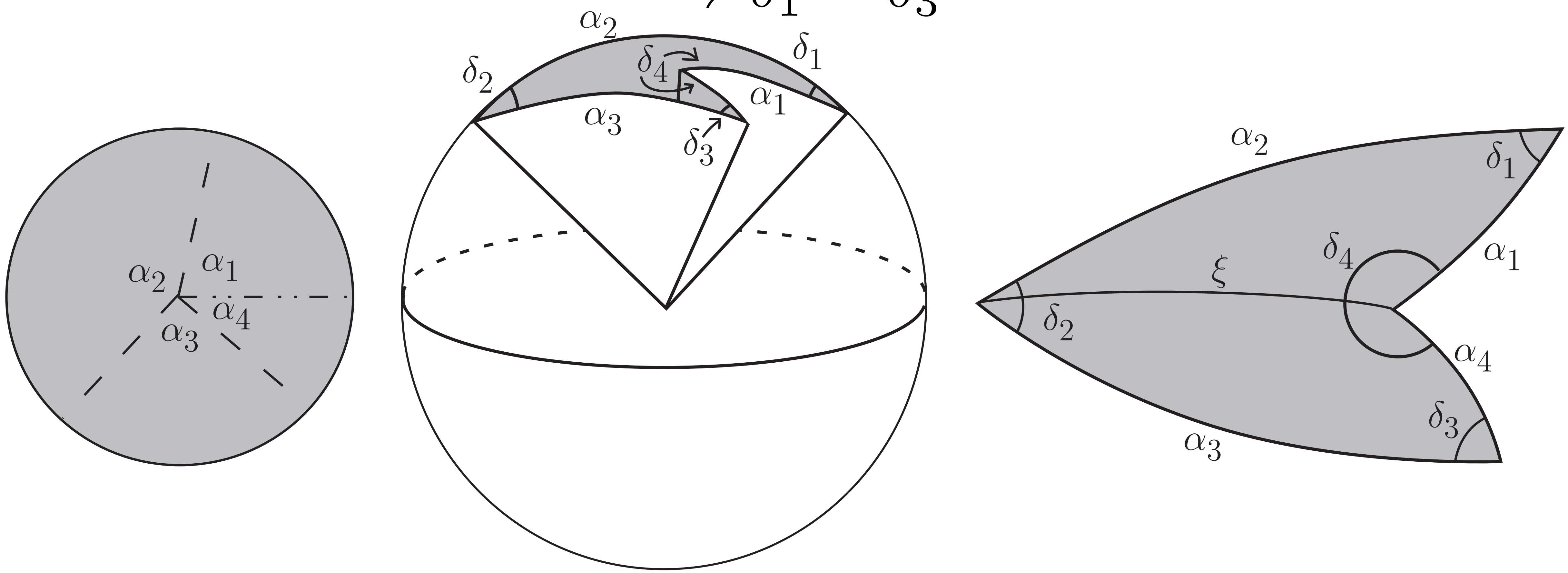
Substituting and setting these equal to each other…

$$\cos\alpha_1\cos\alpha_2+\sin\alpha_1\sin\alpha_2\cos\delta_1=\cos\alpha_1\cos\alpha_2+\sin\alpha_1\sin\alpha_2\cos\delta_3$$

$$\Rightarrow \sin\alpha_1\sin\alpha_2(\cos\delta_1-\cos\delta_3)=0$$

$$\Rightarrow \cos\delta_1-\cos\delta_3=0$$

$$\Rightarrow \delta_1=\delta_3$$



Similarly, $\delta_2=2\pi-\delta_4$

Special degree 4 case!

So we have these great relationships between the folding angles of a degree 4 flat-foldable vertex:

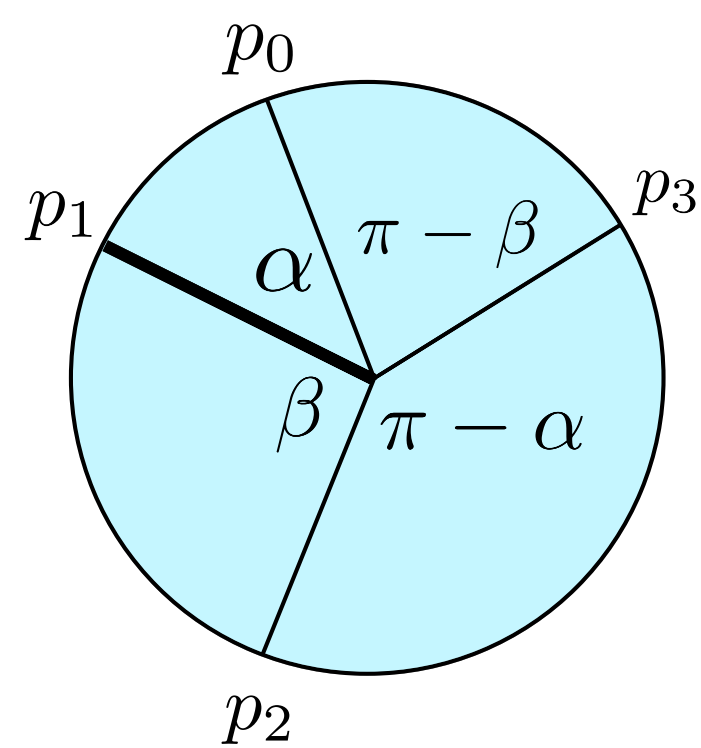
$$\delta_1 = \delta_3 \qquad\qquad \delta_2 = 2\pi - \delta_4$$

so, $p_1 = p_3$ and $p_2 = -p_4$

Next, what can the matrix product tell us?

$$R(L_1)R(L_2)R(L_3)R(L_4) = I$$

$$\frac{\tan\frac{p_3}{2}}{\tan\frac{p_2}{2}} = \frac{1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}{1+\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}$$



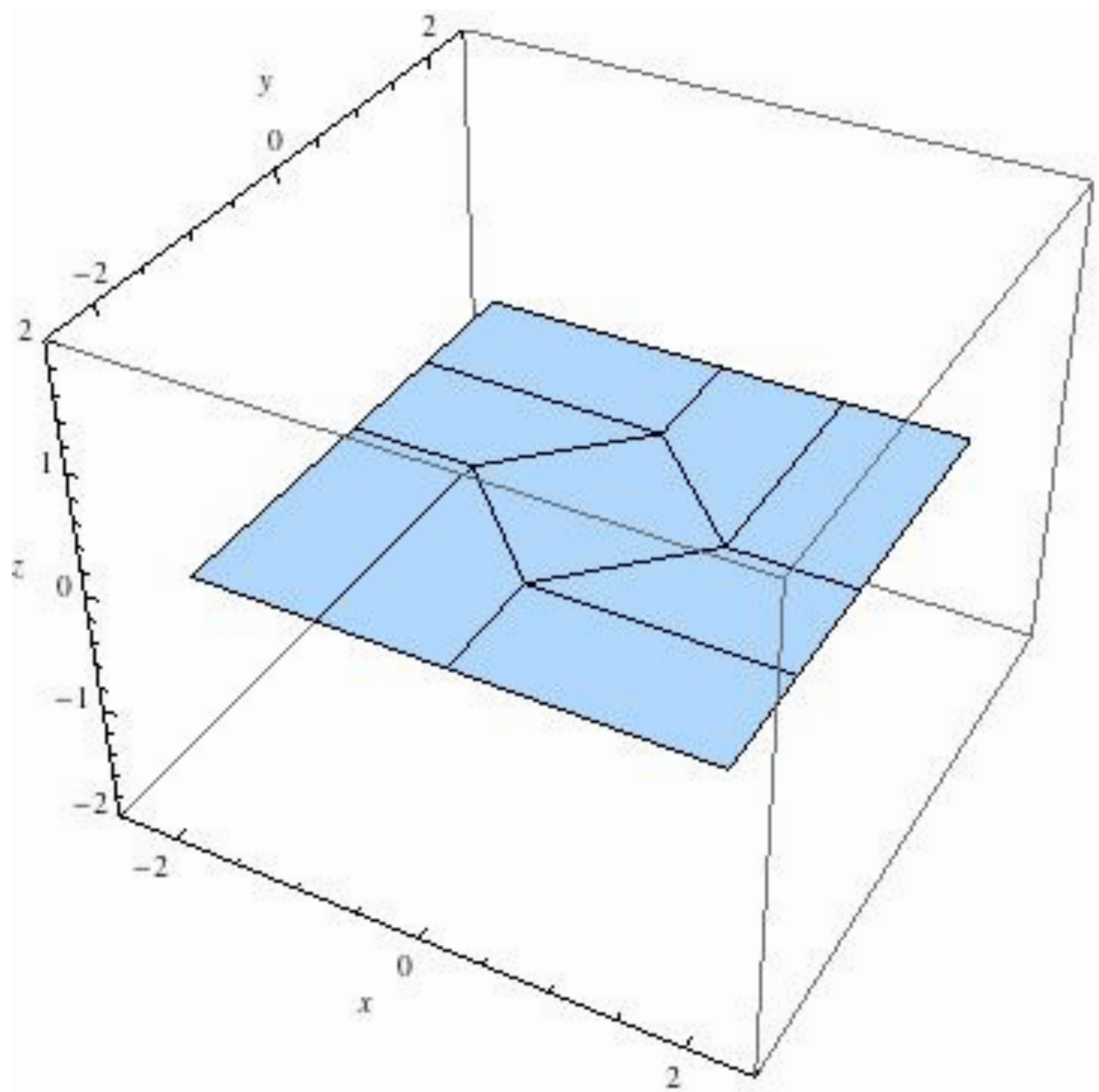
So, $$p_3 = 2\arctan\left(\frac{1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}{1+\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}\tan\frac{p_2}{2}\right)$$

Here $p_0 = p_2$ and $p_1 = -p_3$

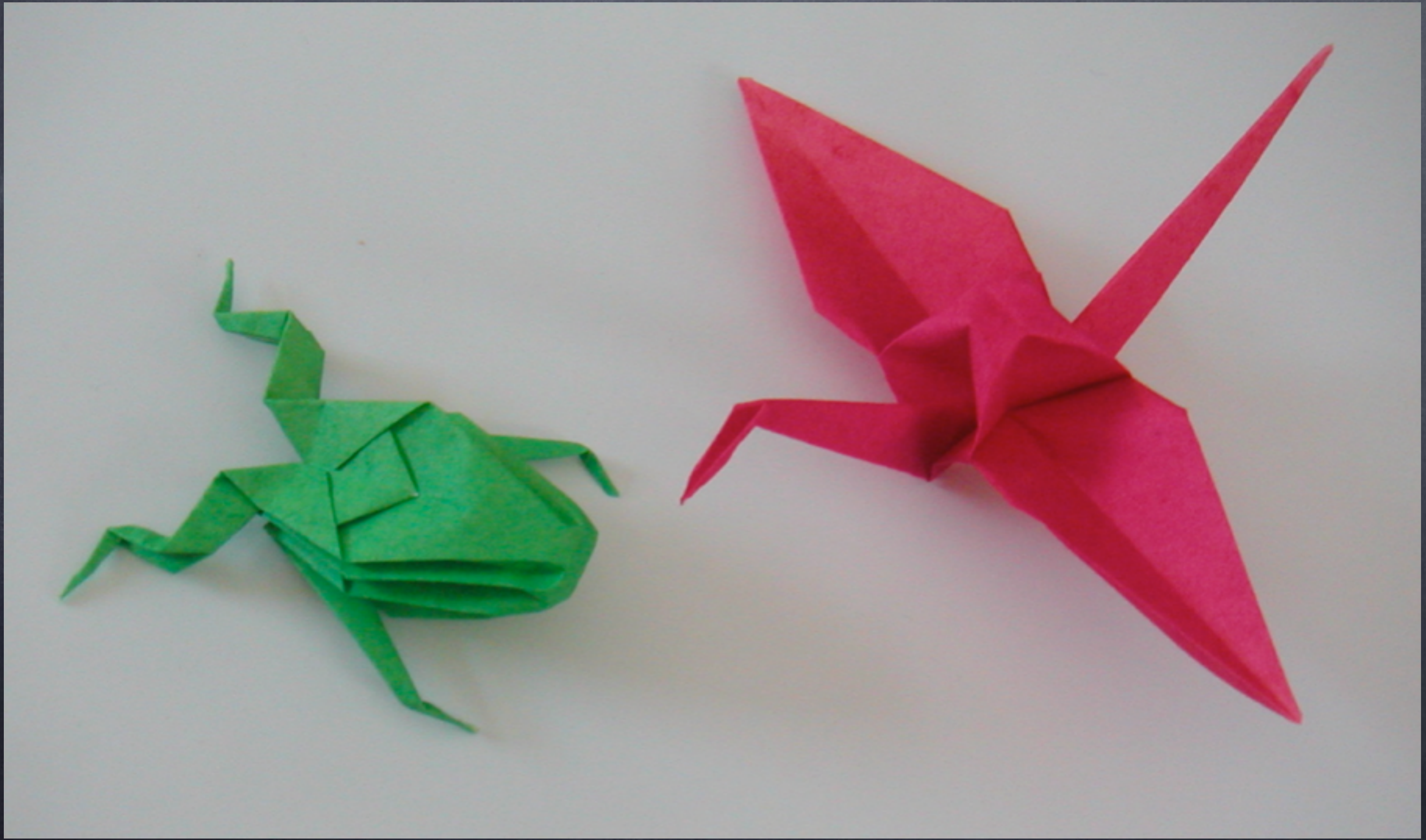
Special degree 4 case!

Another example of an rigid fold modeled by matrix transformations:

HCSSiM 2008 student
Ben Kraft's
square twist animation
via matrix algebra



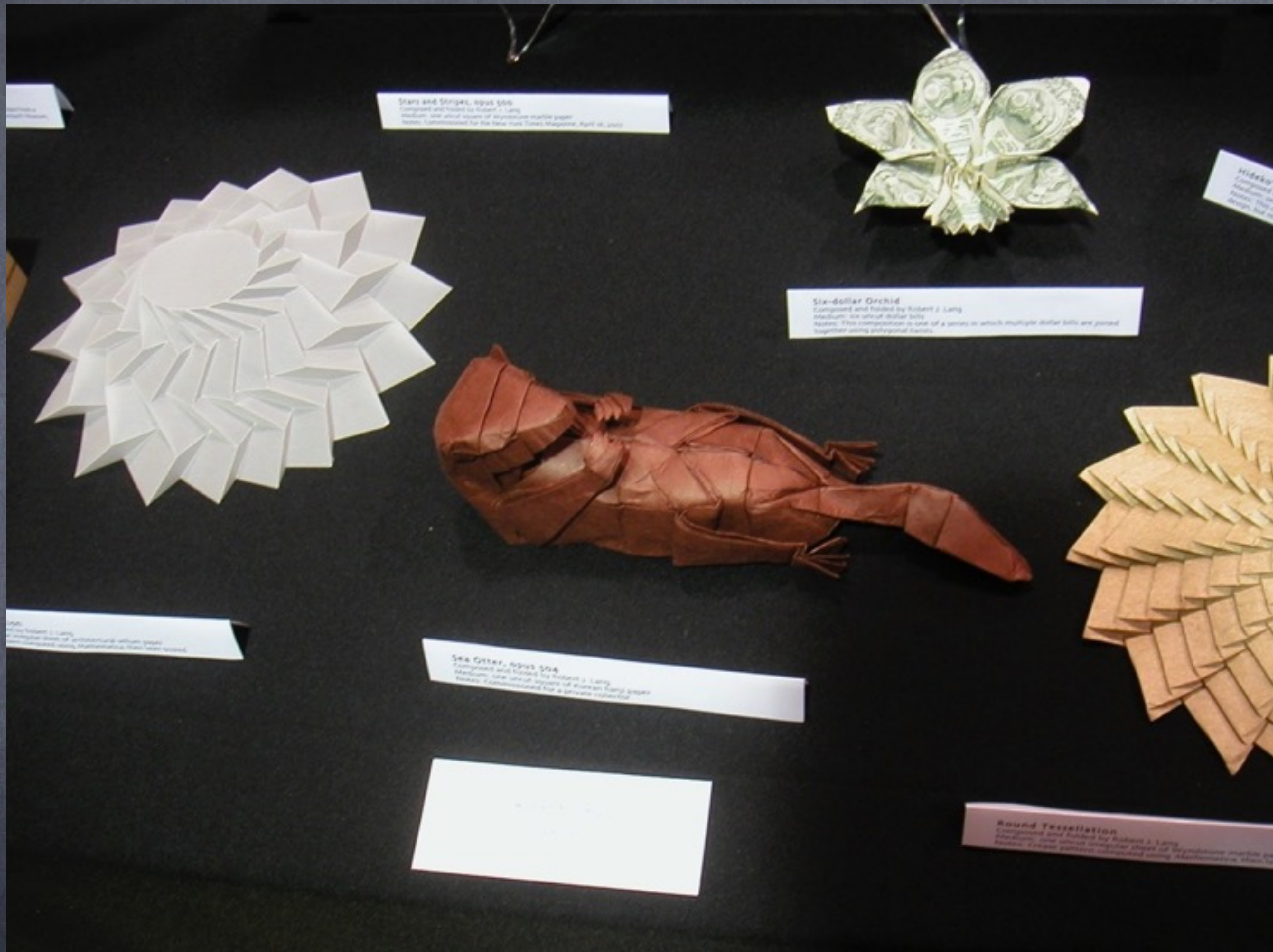
Classic (traditional) Origami



Origami today

Robert Lang's Black Forrest Cuckoo Clock, 1987

OUSA Convention 2007



Robert Lang's Sea Otter

OUSA Convention 2007

Hojyo Takashi

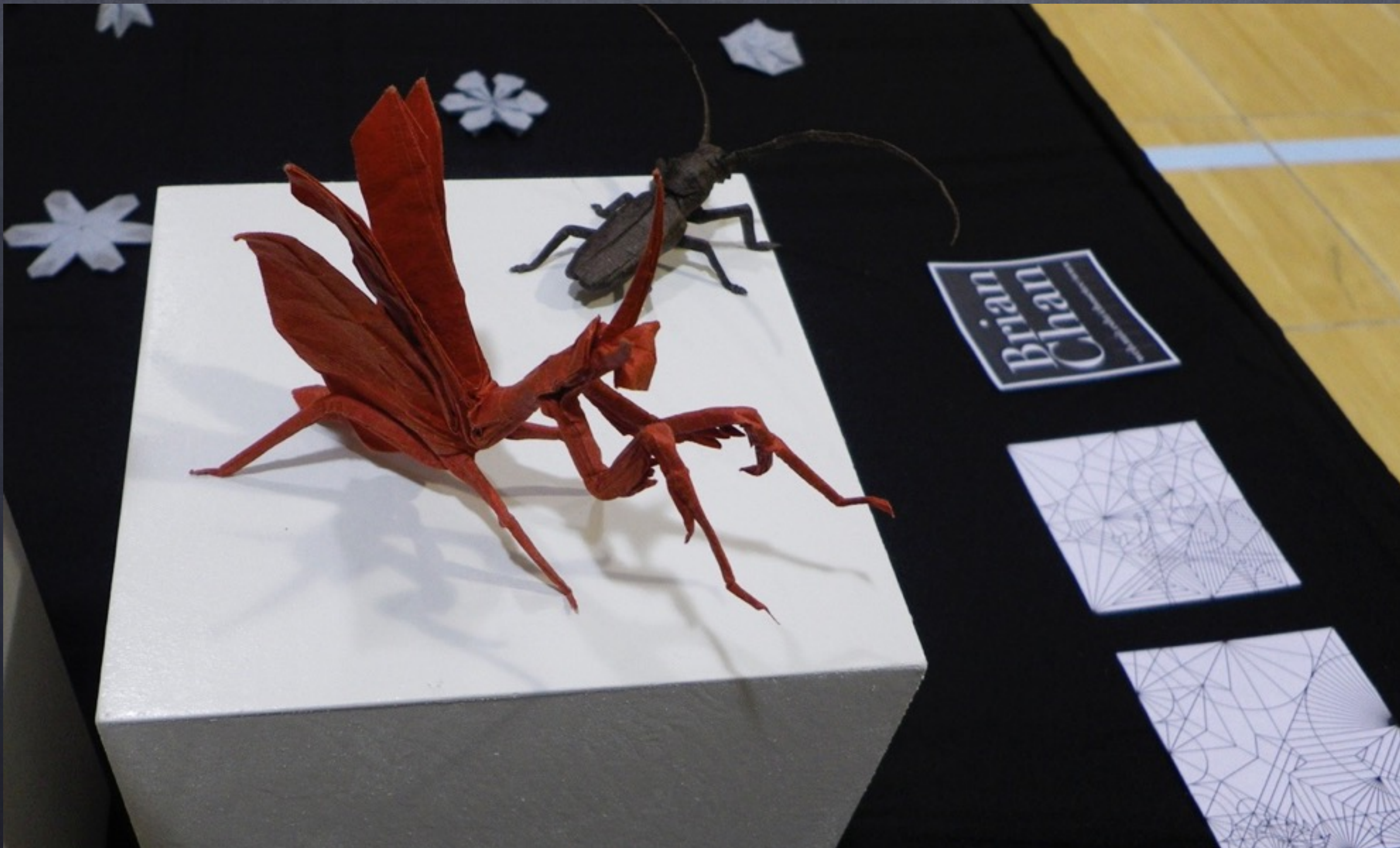
OUSA Convention 2007

Satoshi Kamiya

OrigamiUSA Convention 2011

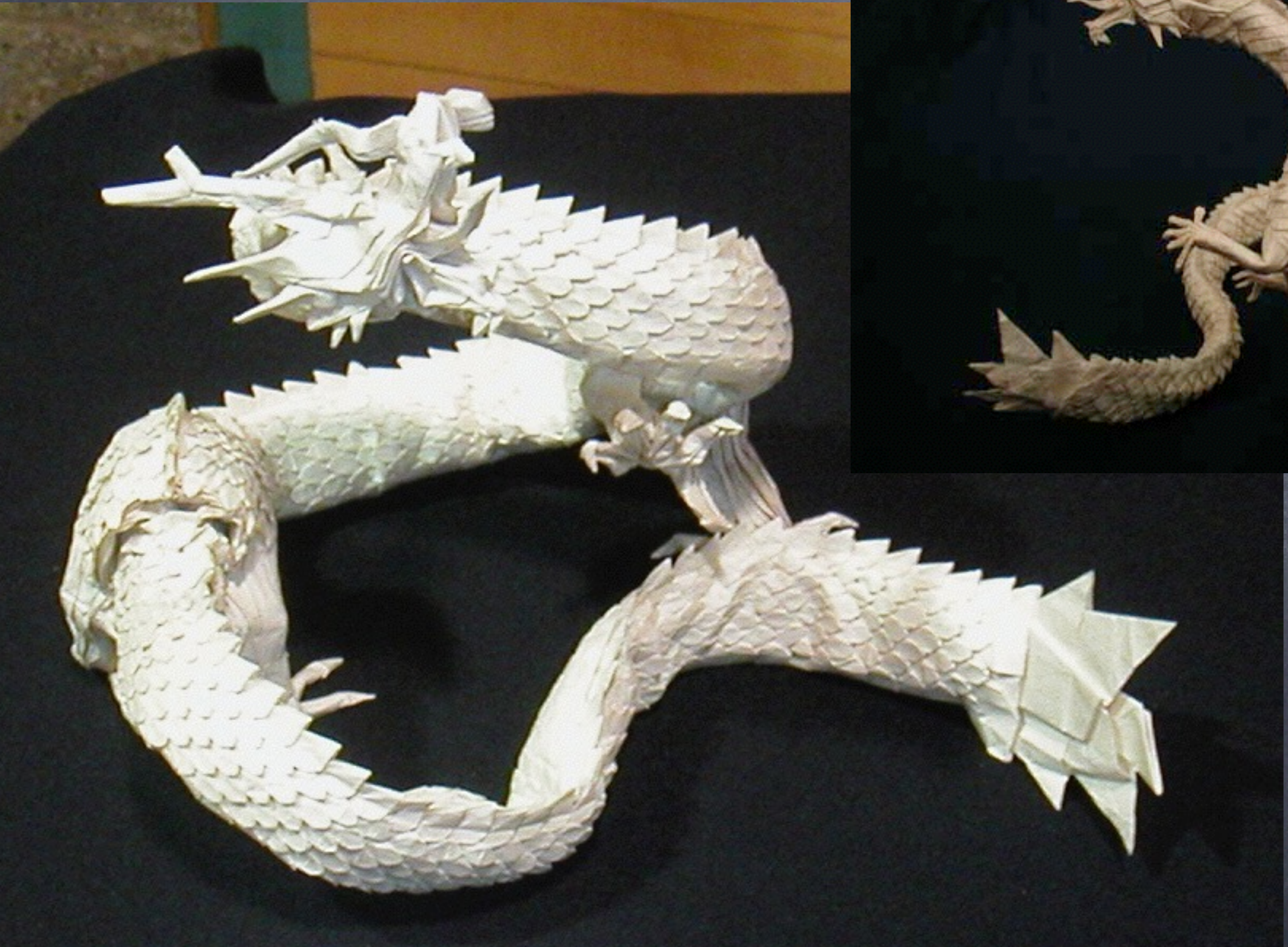
Praying Mantis, Brian Chan, 2011

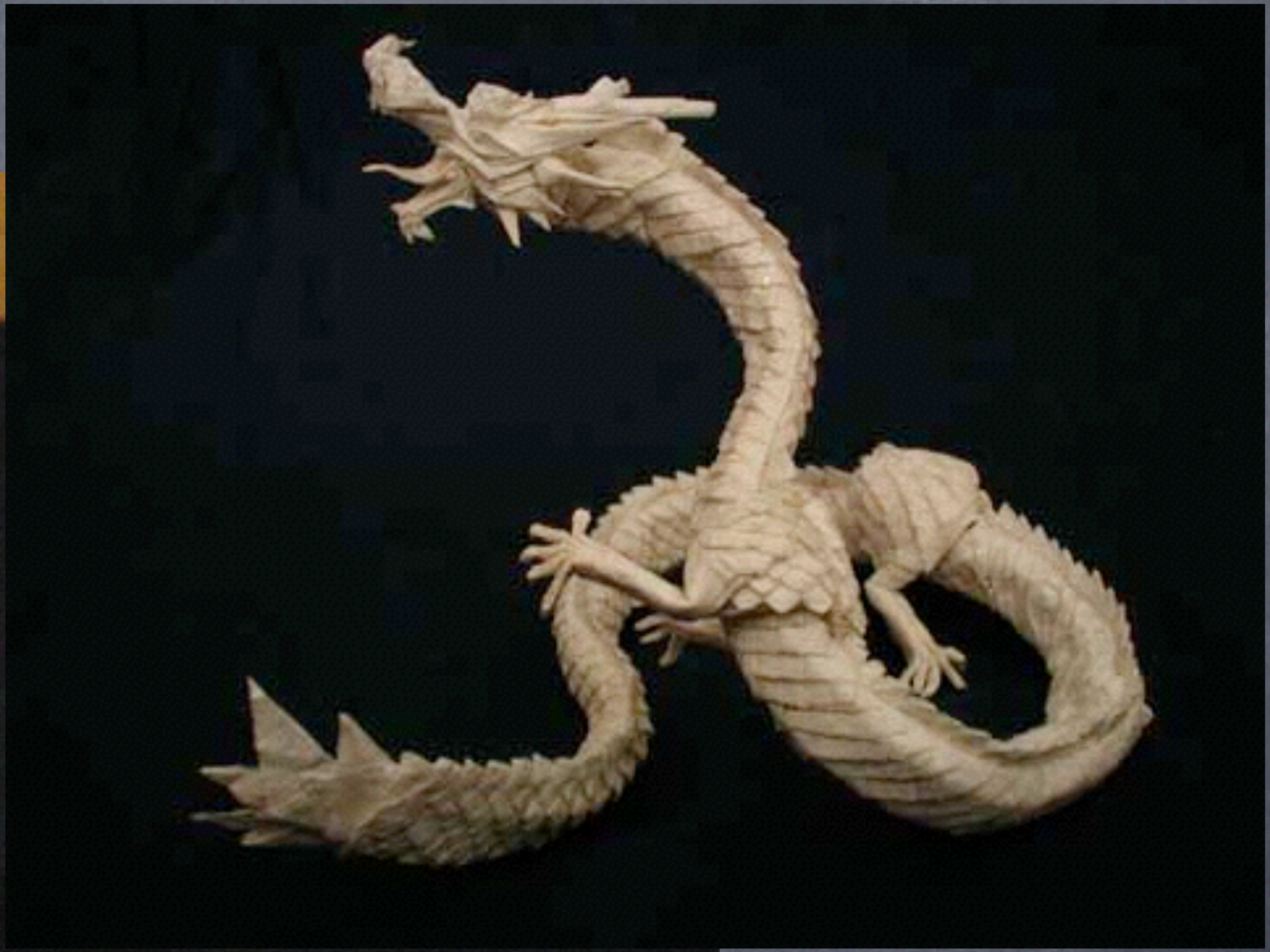
OrigamiUSA Convention 2011



Praying Mantis, Brian Chan, 2011

Satoshi Kamiya's Ryu-Zin

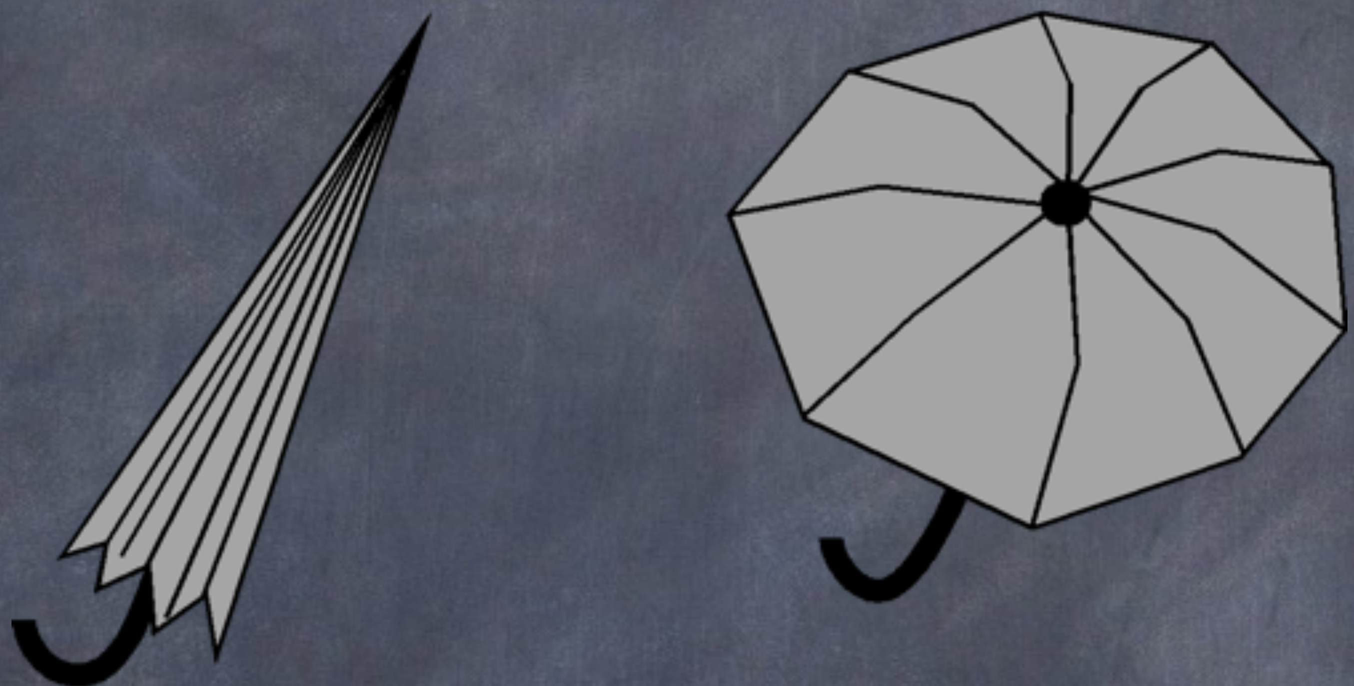




OUSA Convention 2007

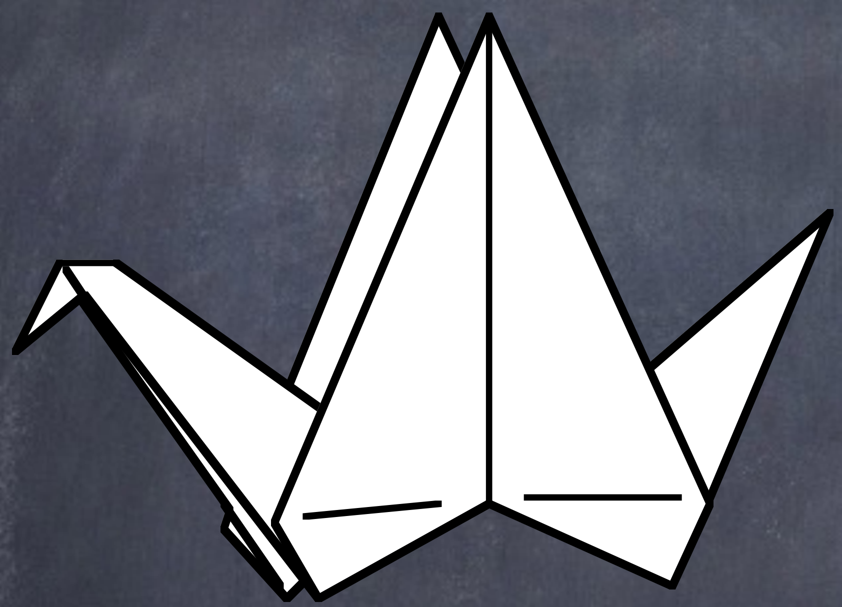
Joseph Wu

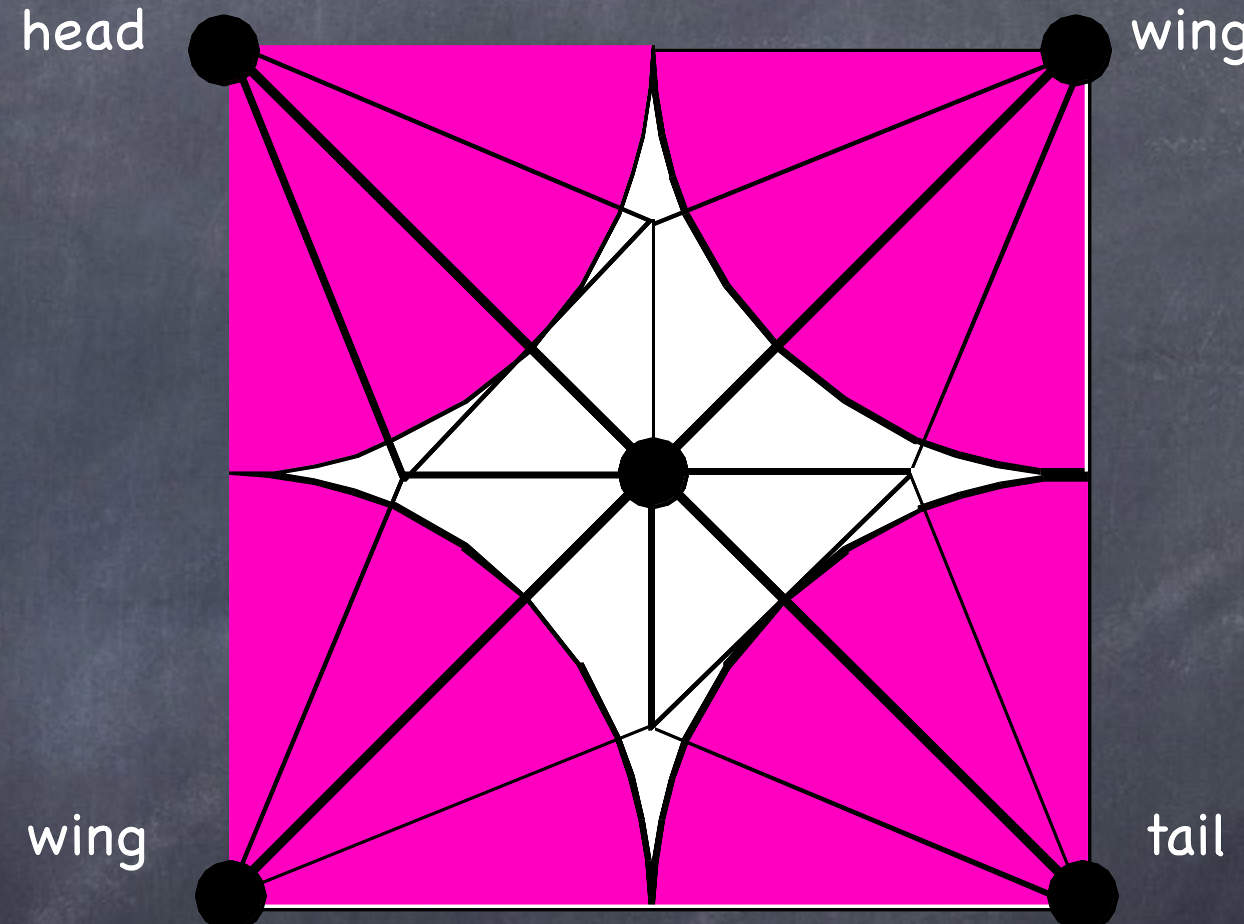
Appendages in origami are like umbrellas



each appendage in the model will
require a circle of paper

Example: the crane





Thus origami design is related to circle packing.

Credits: Robert Lang, Jun Maekawa, Toshiyuki Meguro, Fumiaki Kawahata

Example: how to fold a sea urchin!

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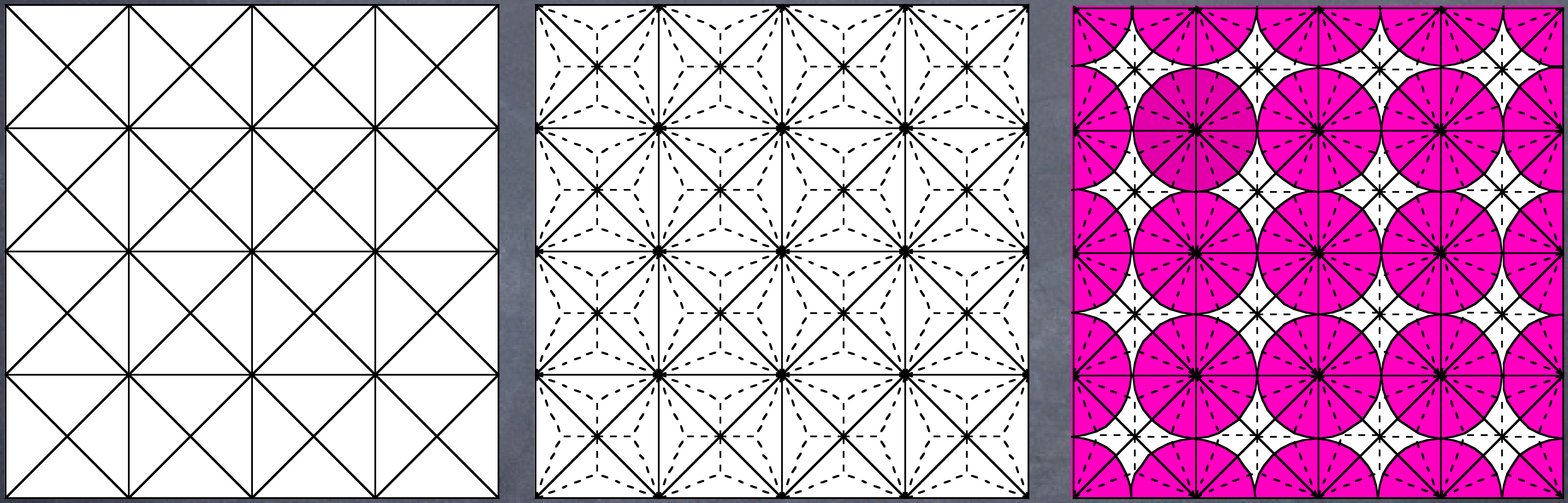

Start with a grid of right triangles and rabbit ear!

Jun Maekawa's Demon (1983)

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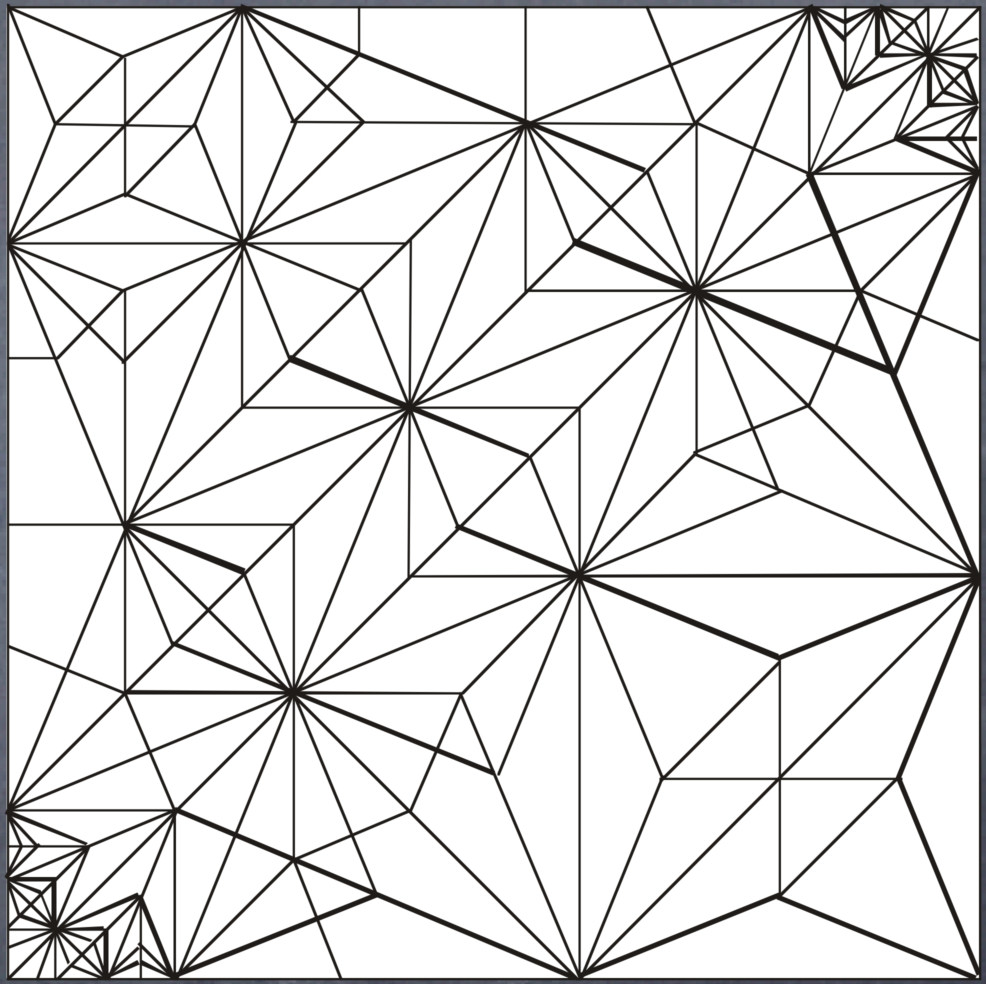
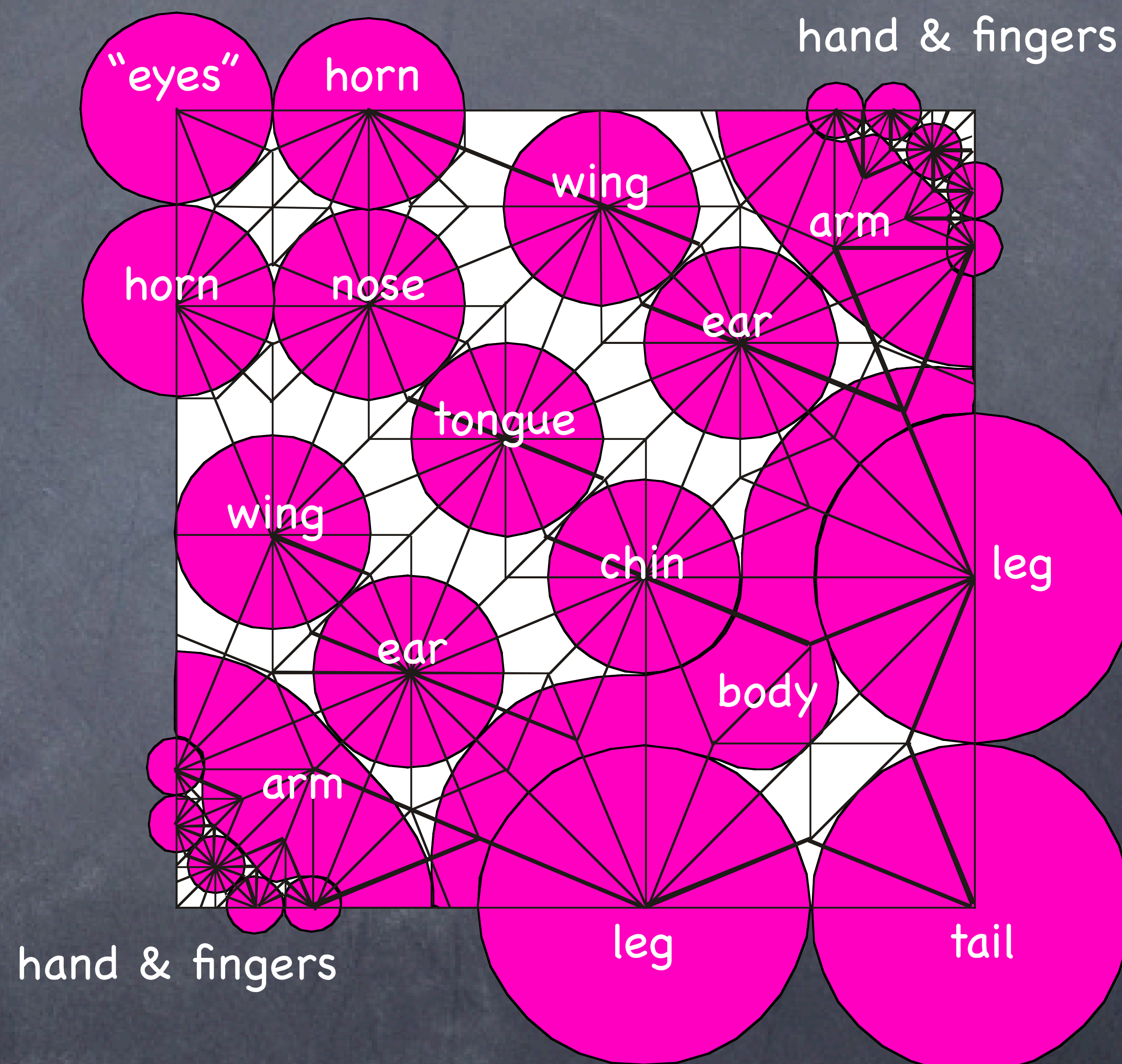


Jun Maekawa's Demon (1983)



Satoshi Kamiya's Wasp (2001)

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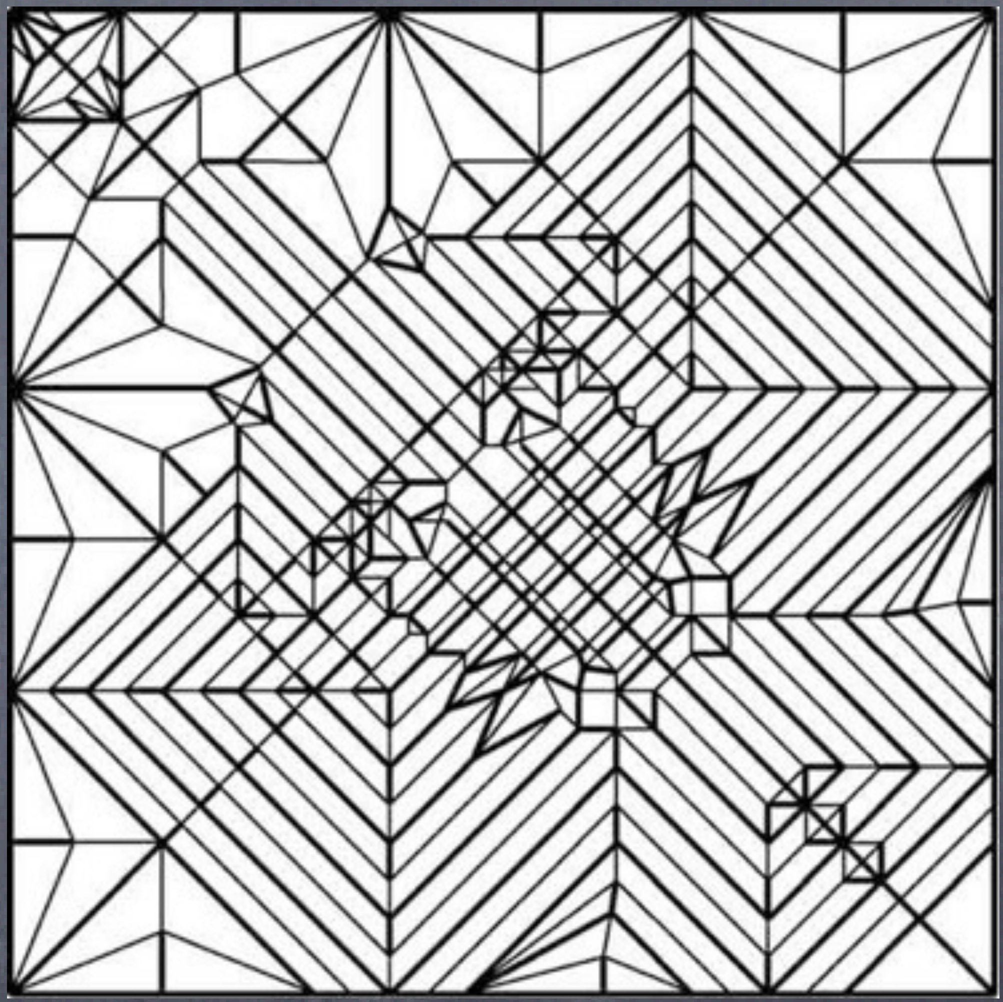

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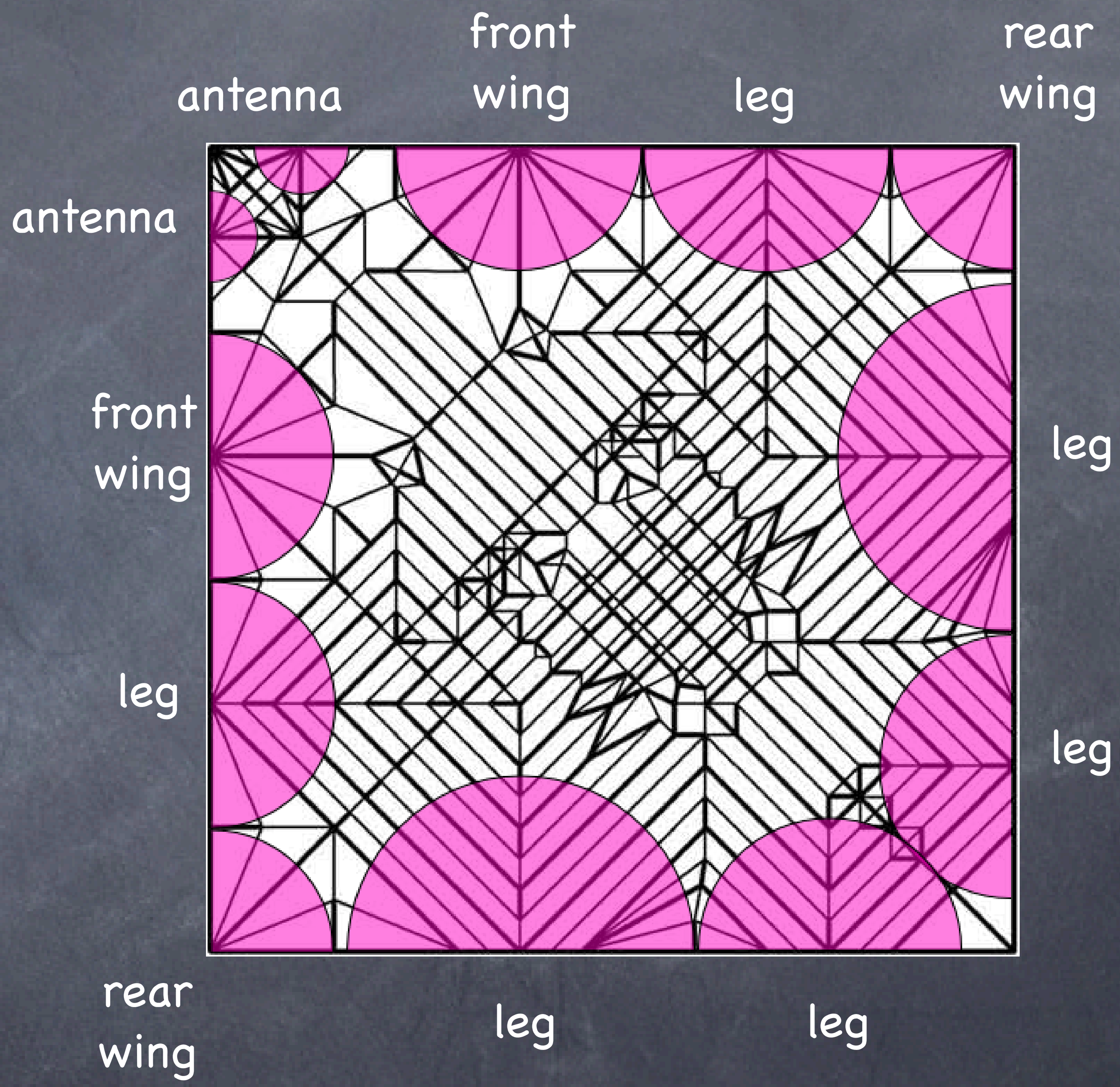
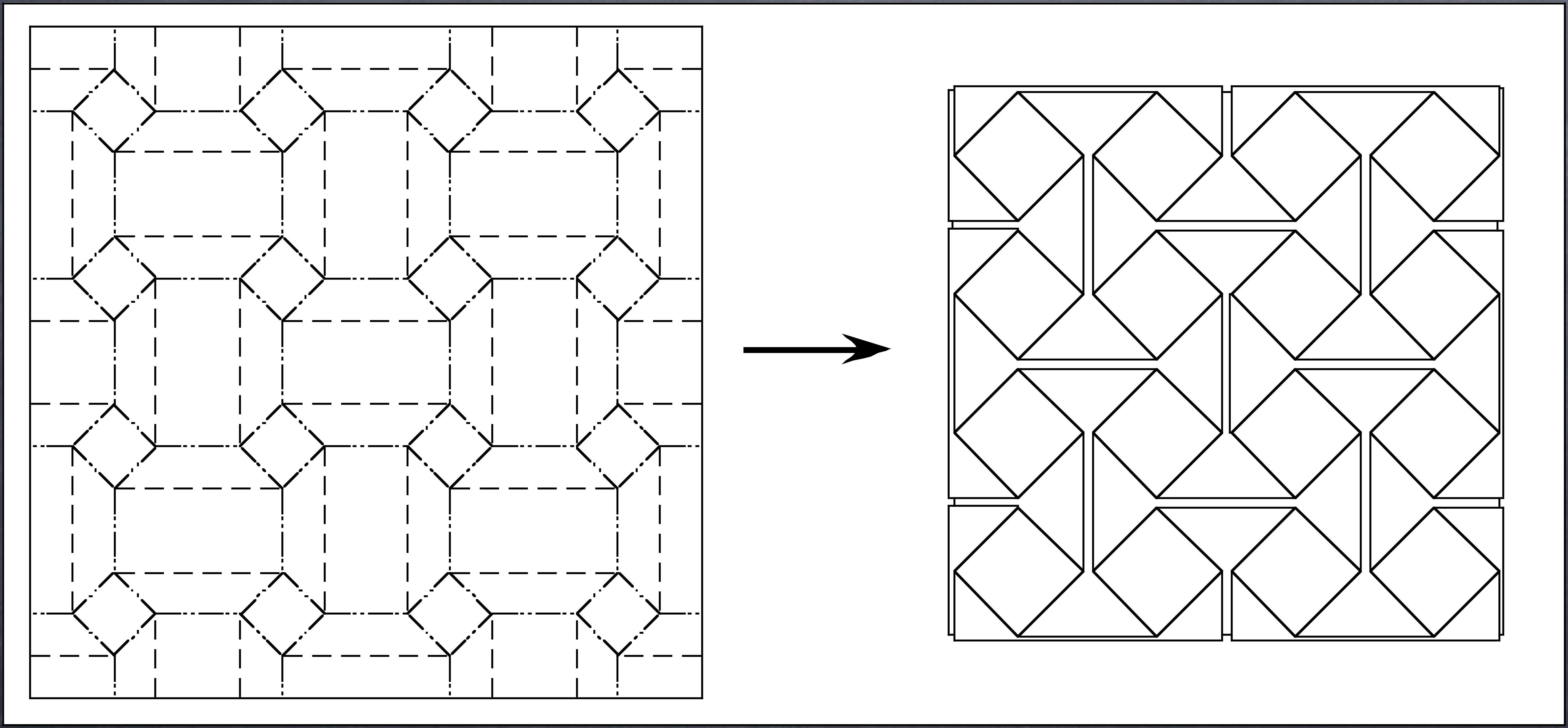
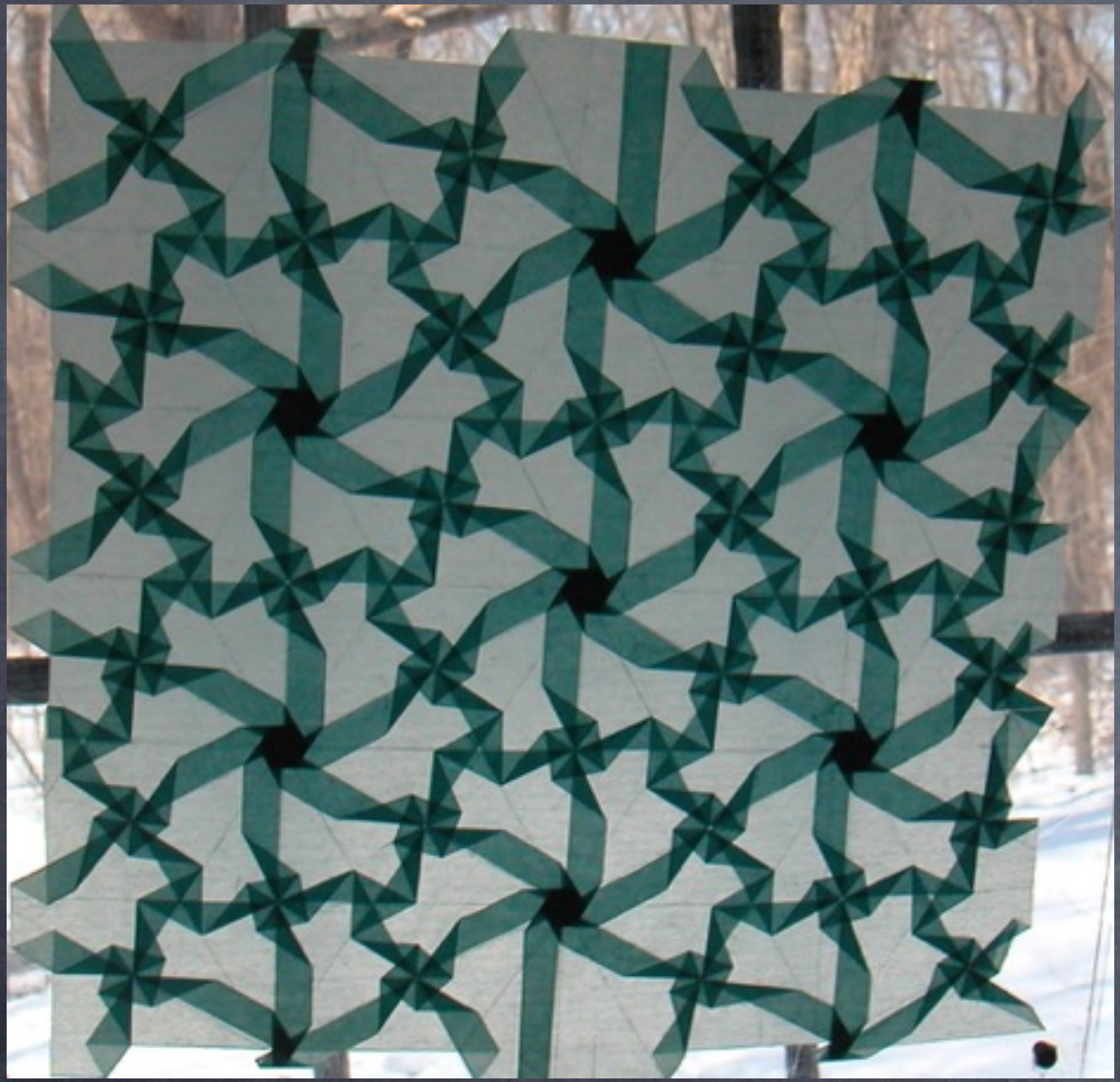


Origami Tessellations

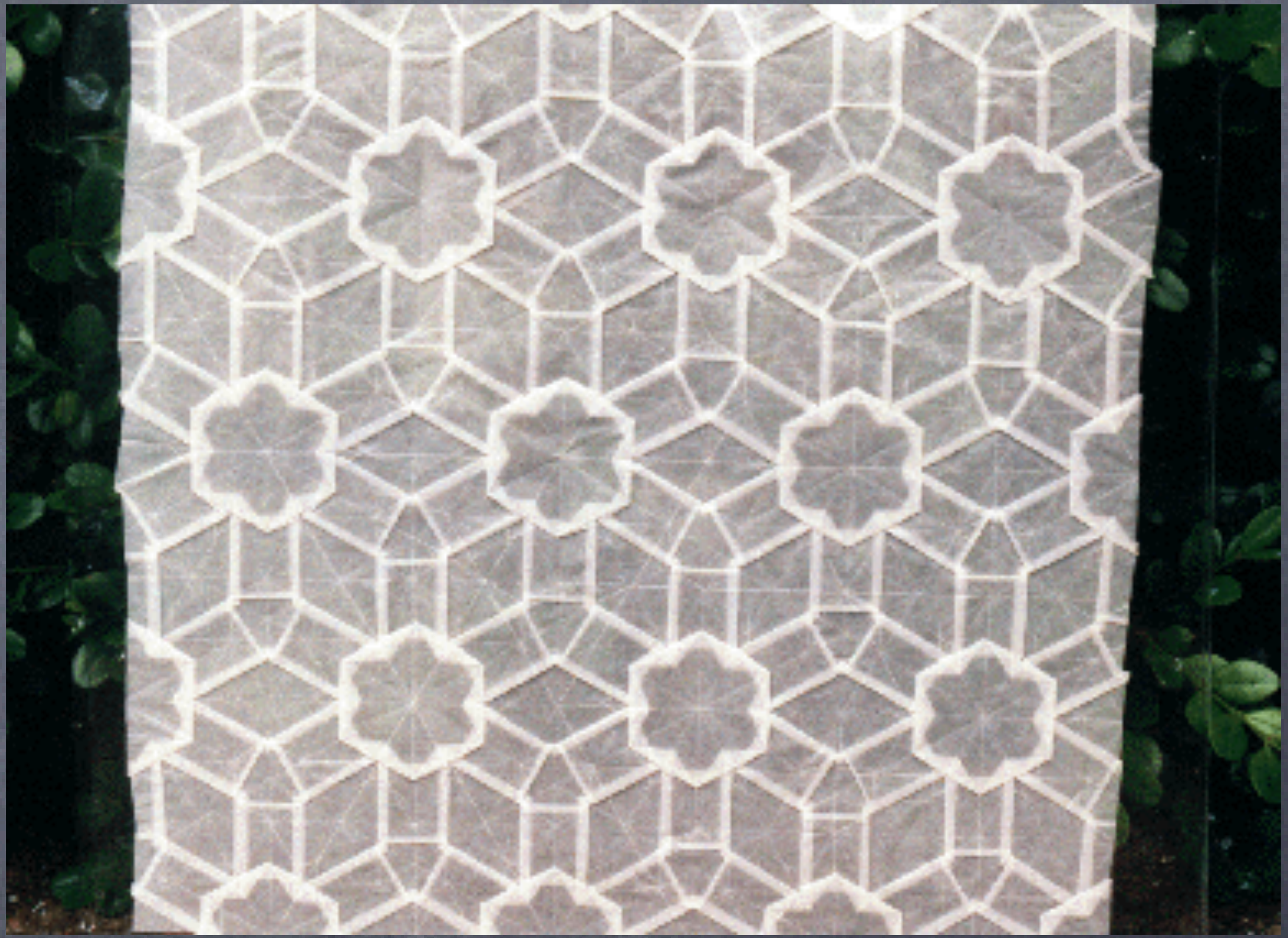


square twist tessellation
(Fujimoto, 1970s)

Origami Tessellations

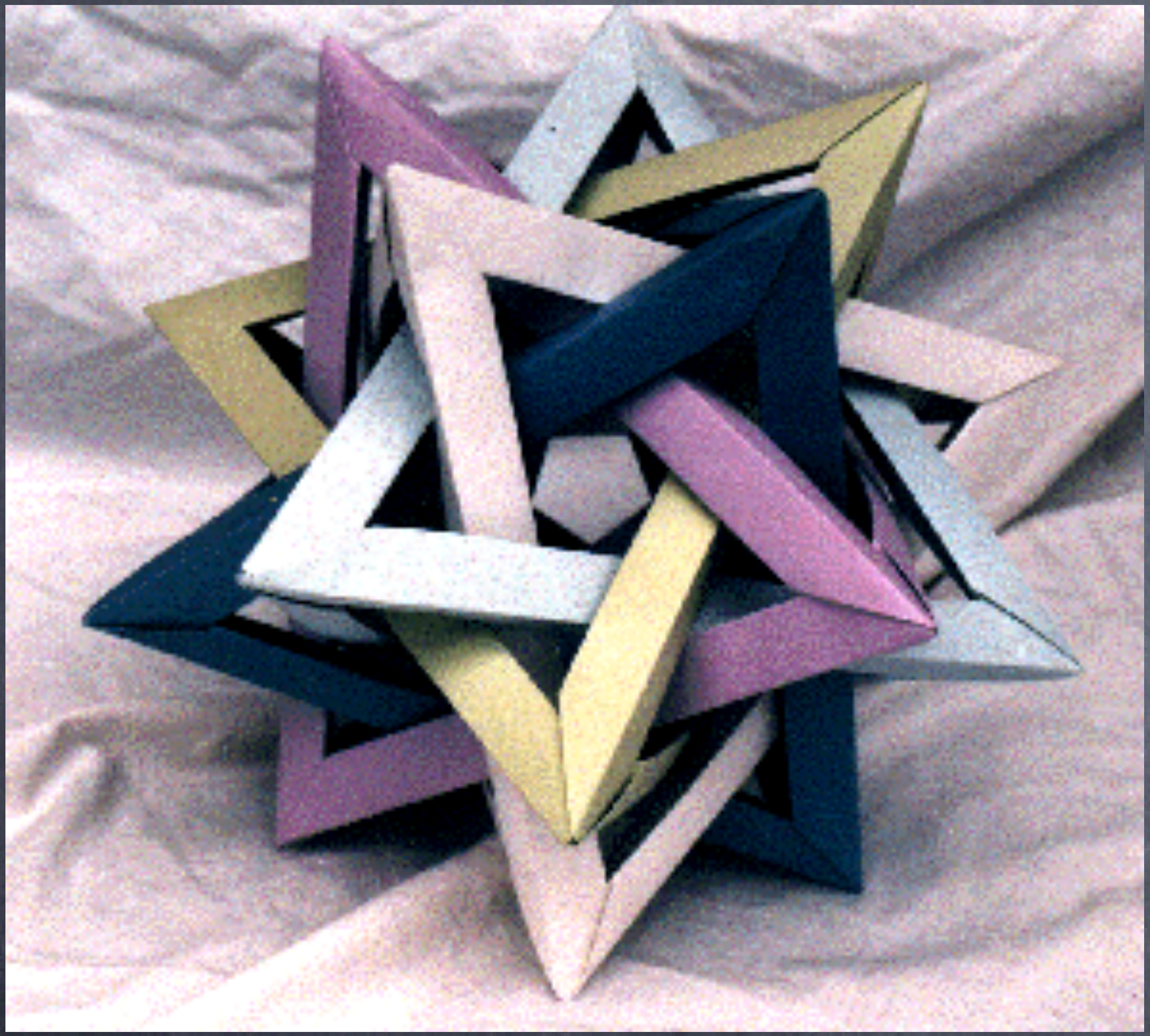


3.4.6.4 tessellation
(Hull, 1993)

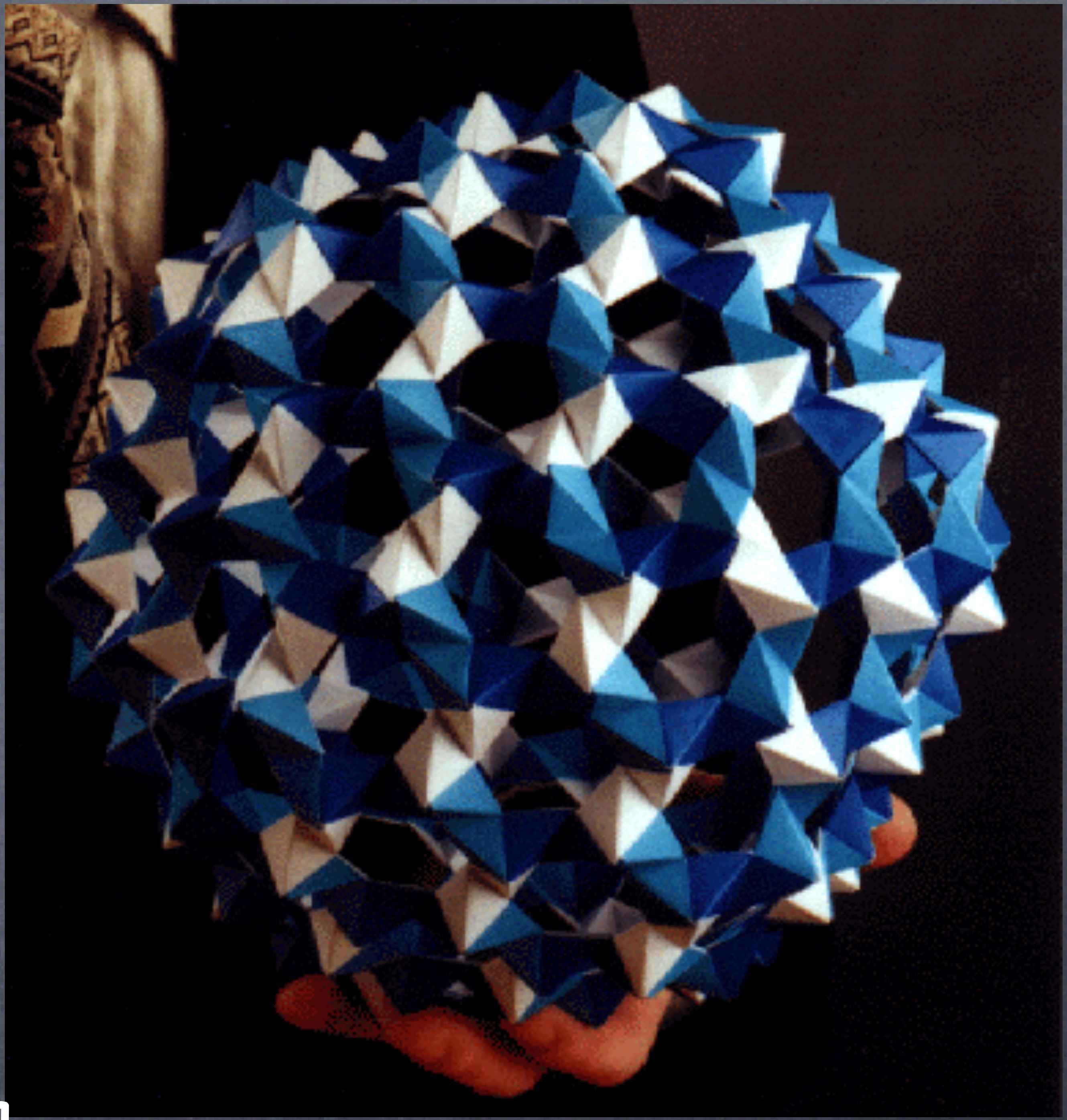


3.6.3.6 tessellation
(Hull, 1993)

Modular Origami

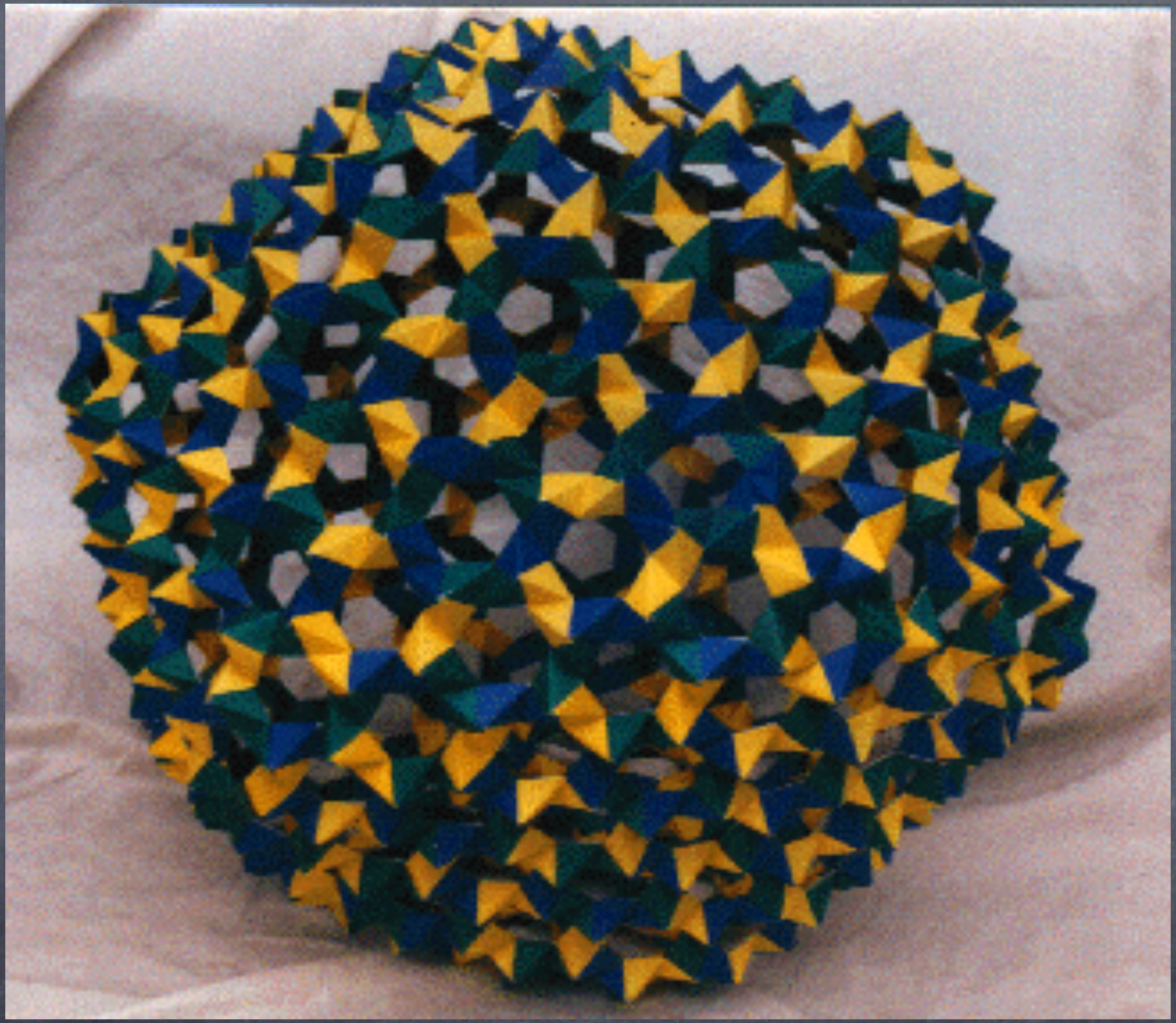


Five Intersecting Tetrahedra
(Hull, Ow, 1996)



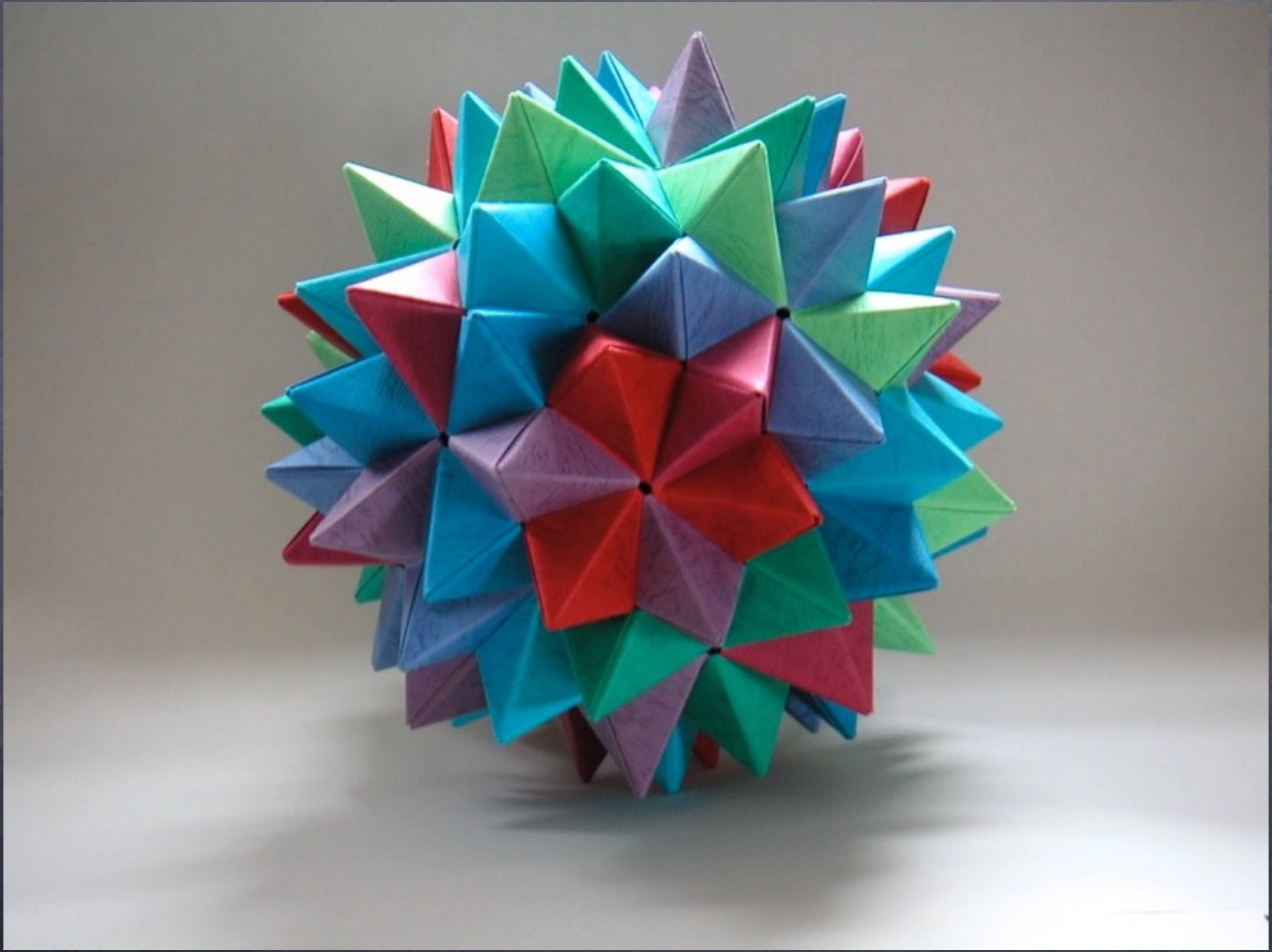
120-unit Buckyball
(Hull, 1994)

Modular Origami



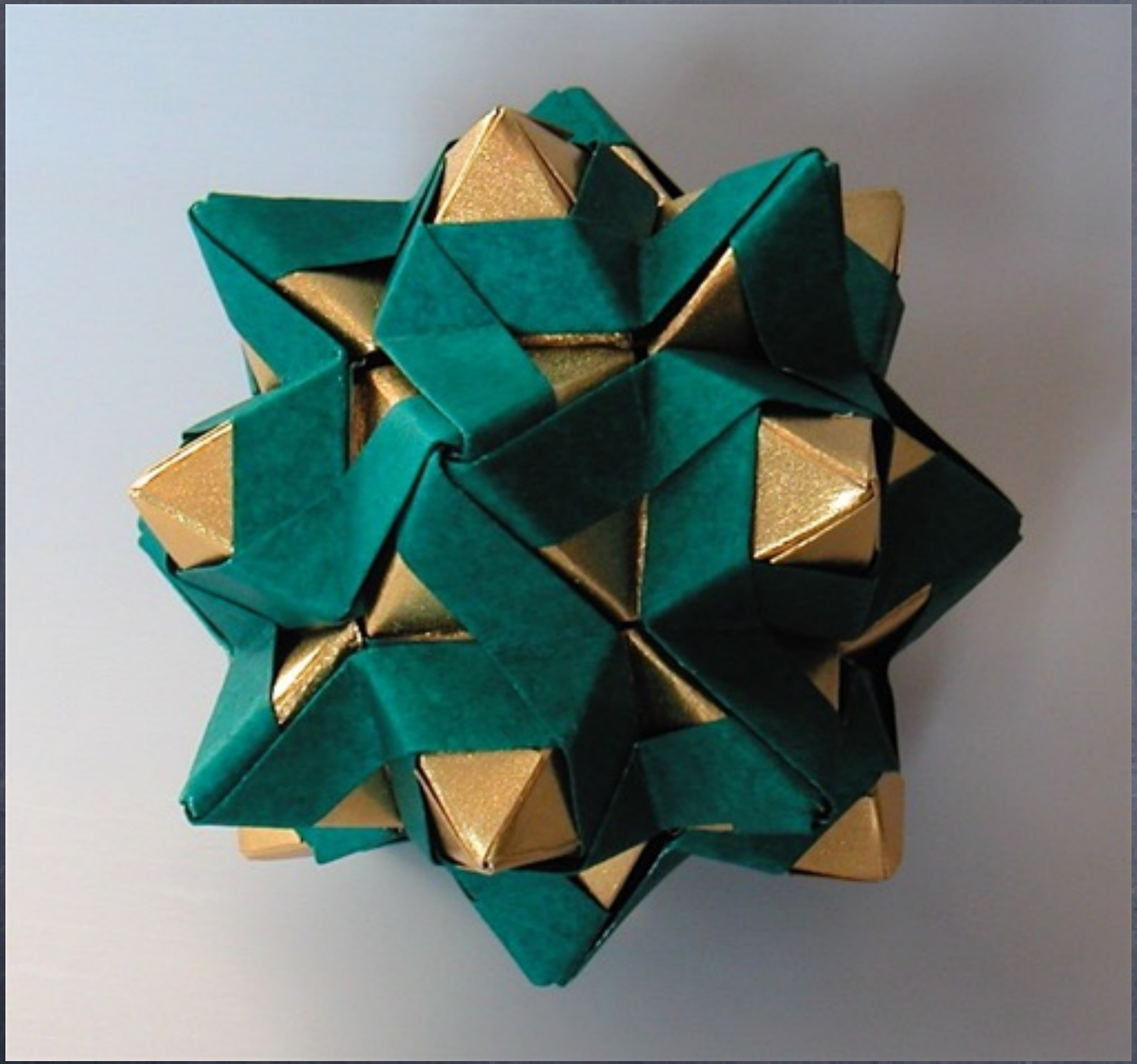
810-unit Buckyball (1994)

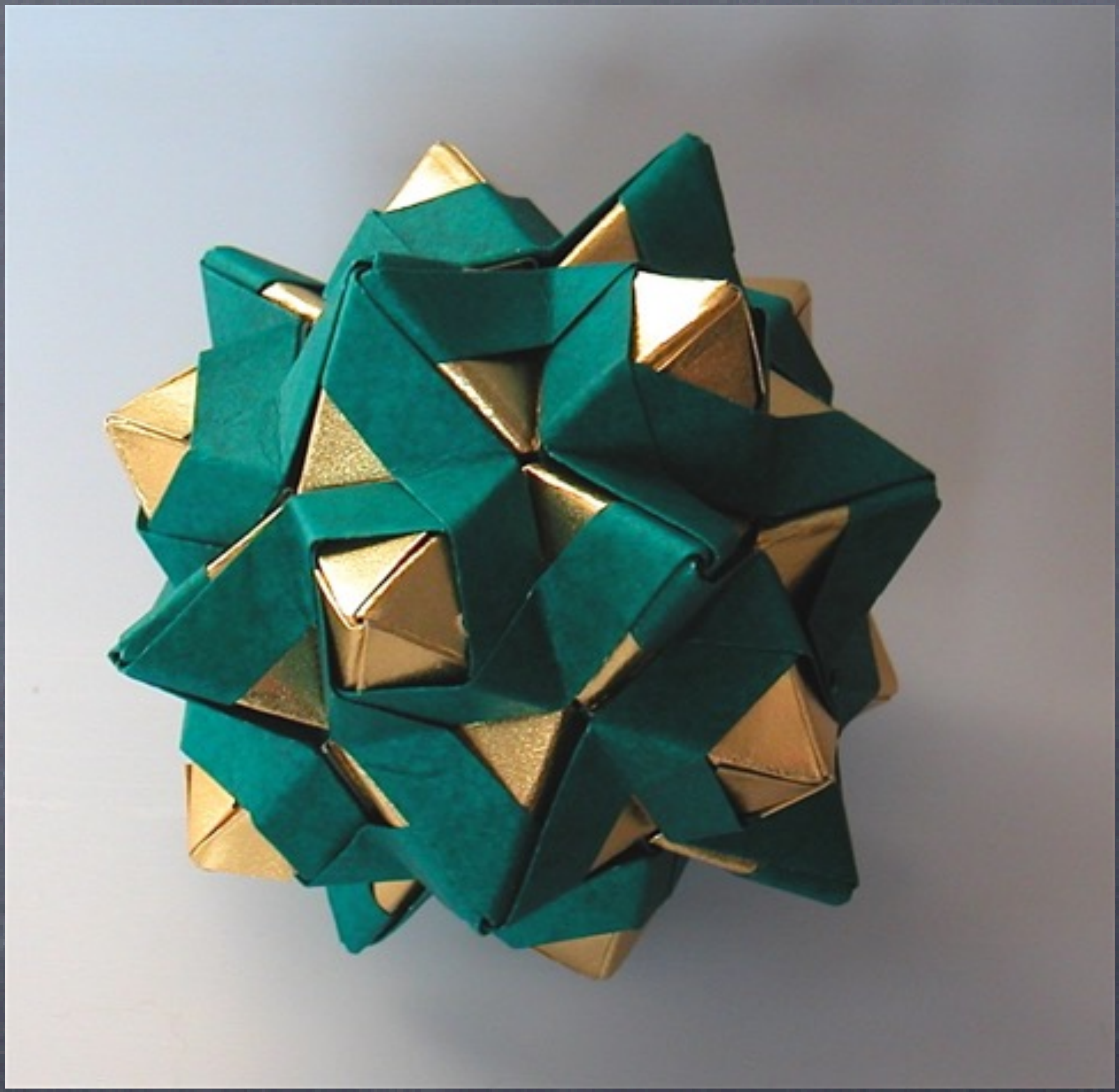
Modular Origami



180-unit Spiked Rhombic Enneacontahedron (2004)

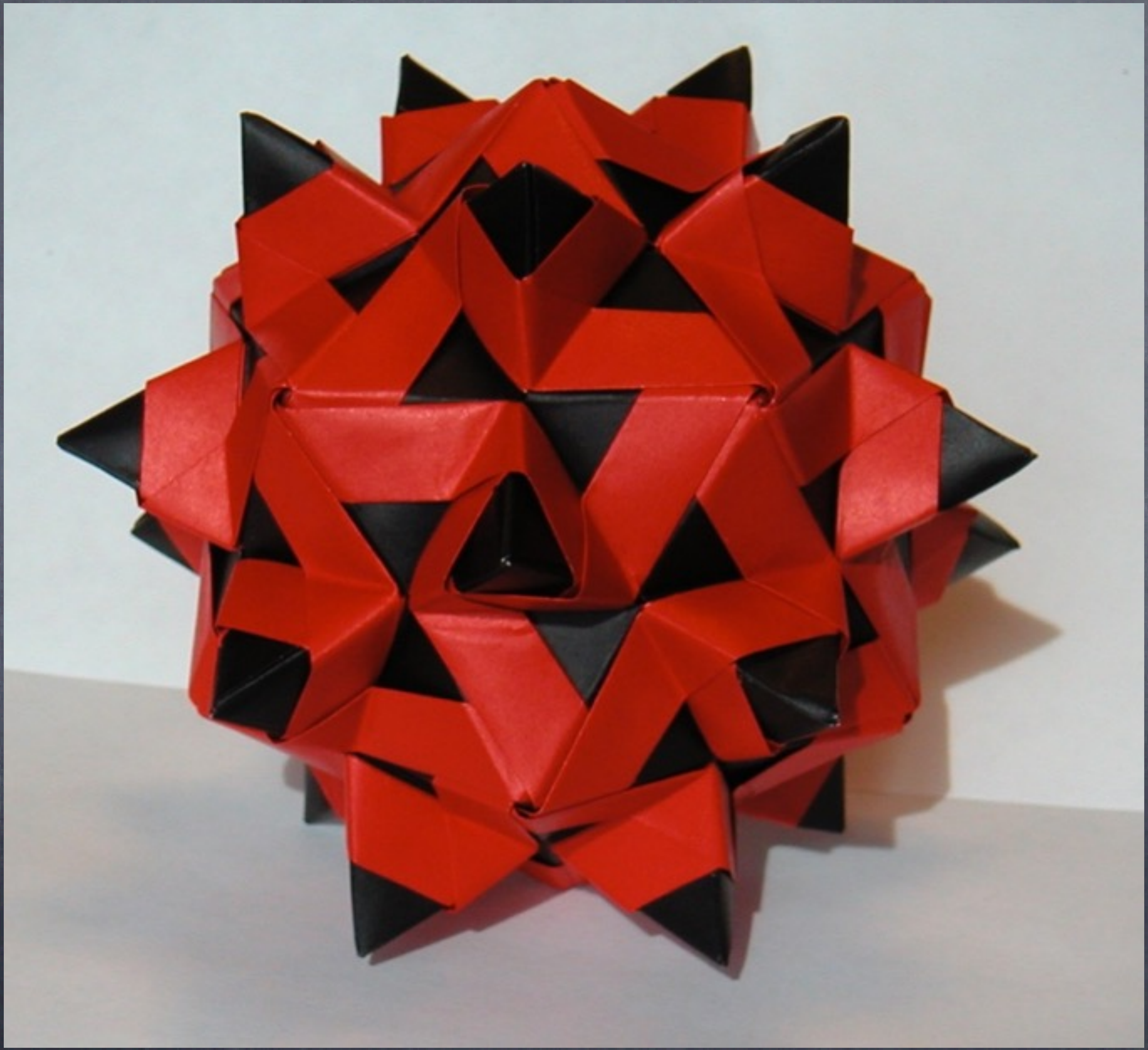
Modular Origami





Spiked Rhomicuboctahedron
made from Hybrid Units (48 units, 2008)

Modular Origami

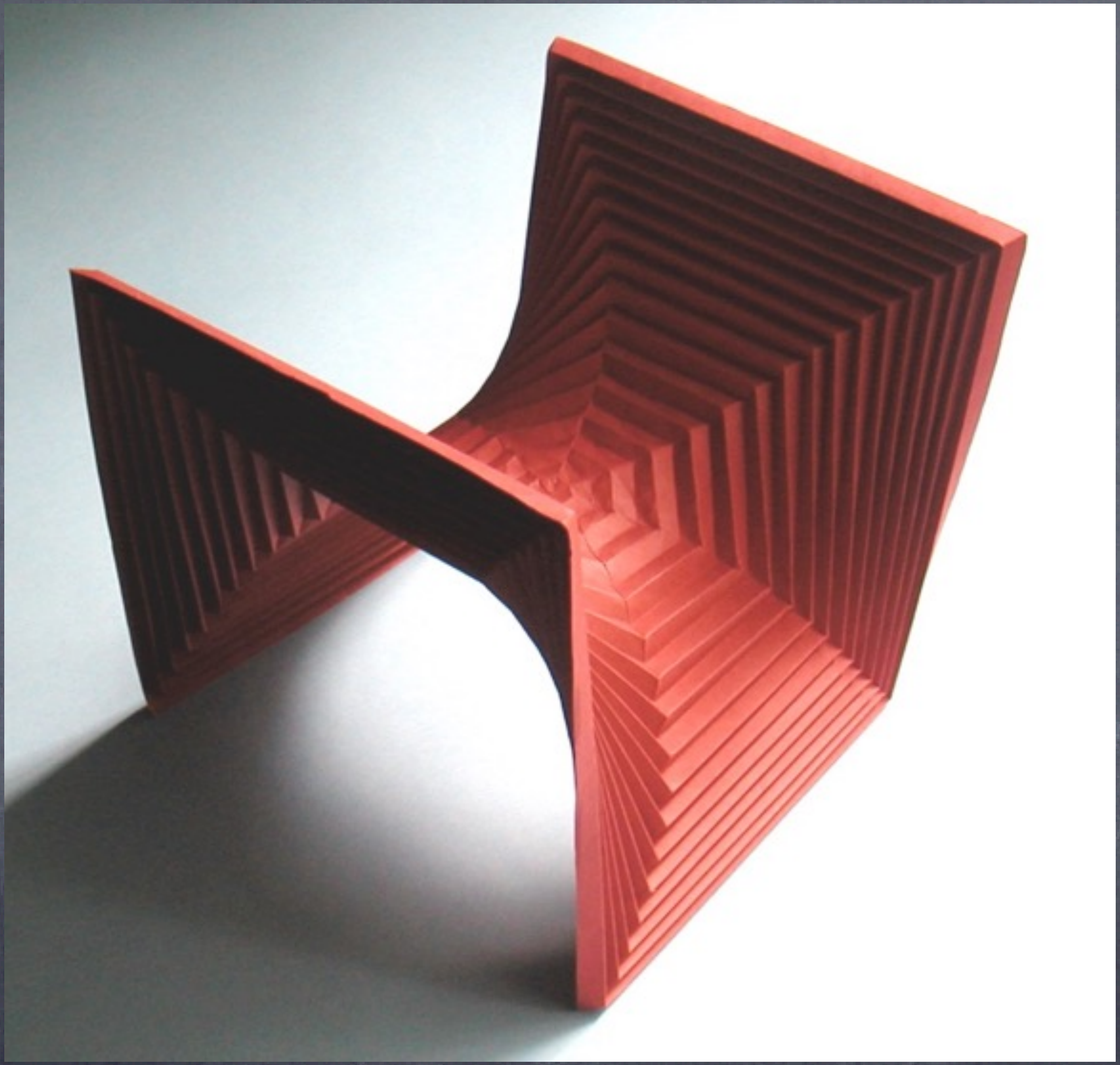


Spiked Icosadodecahedron (60 Hybrid Units, 2008)

Modular Origami

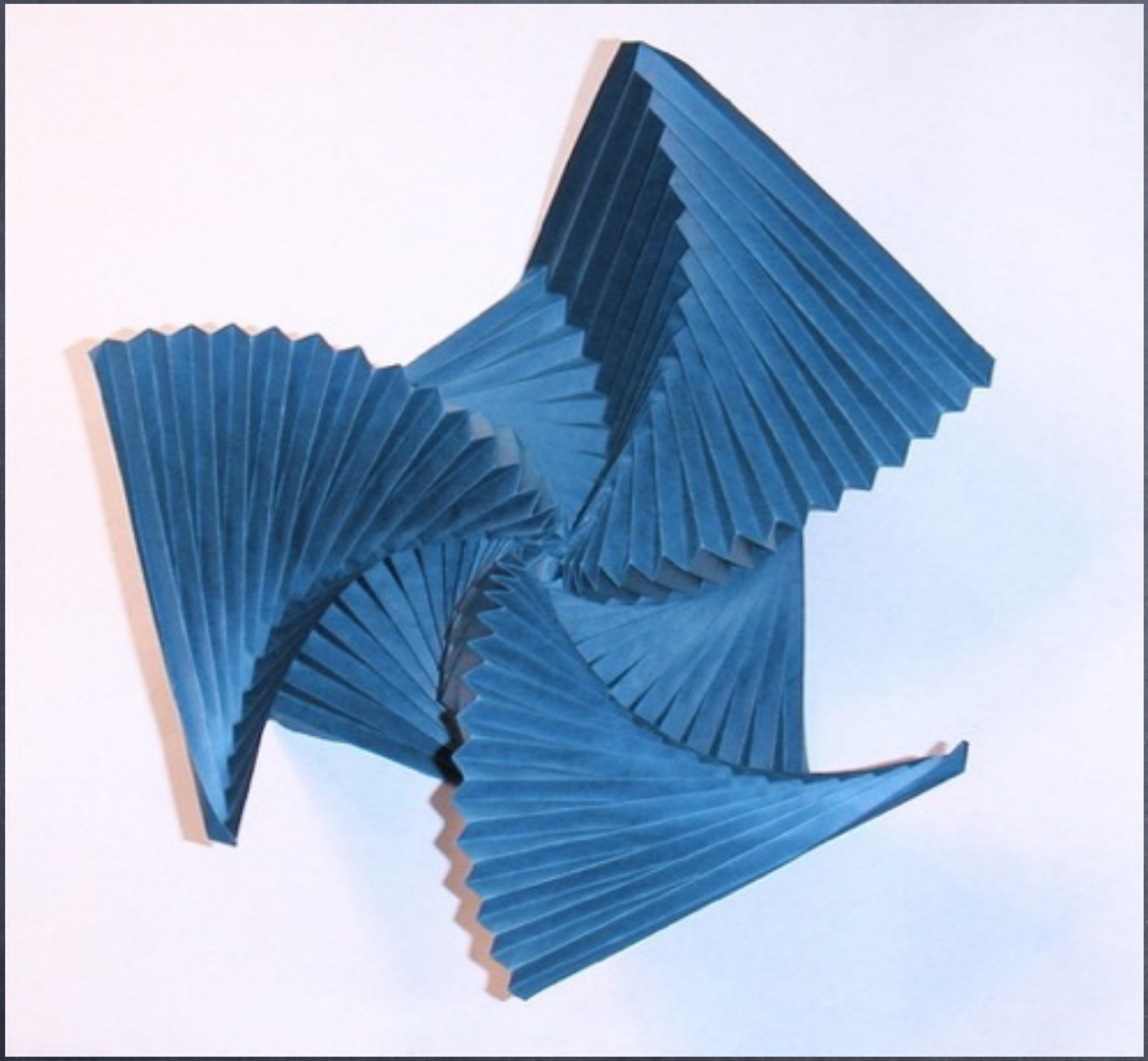
Spiked Rhombicosadodecahedron
(120 Hybrid Units, 2008)

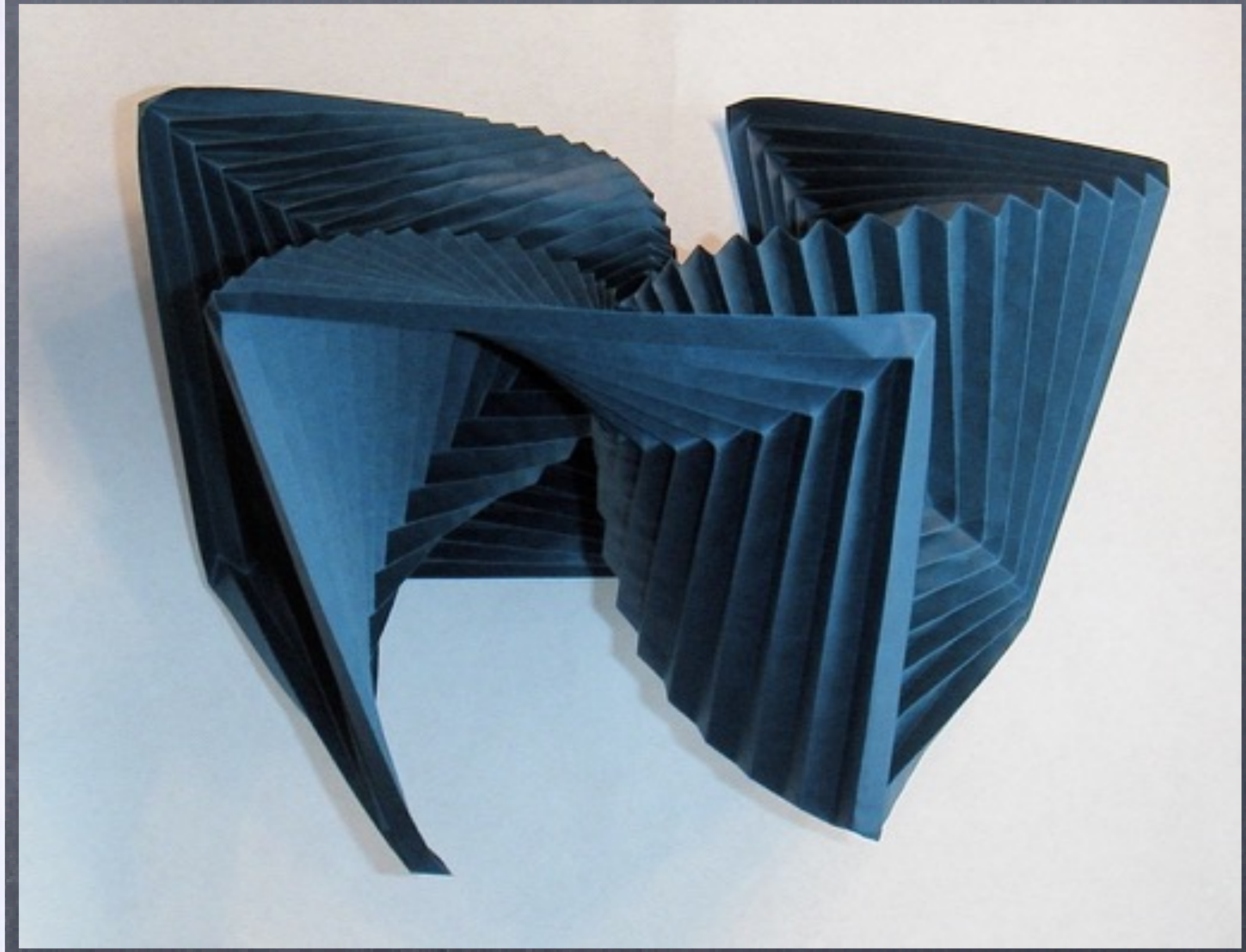
Hyperbolic Cube (2006)



Wet-folded from one 2-foot diameter octagon.

Corrugation Folds

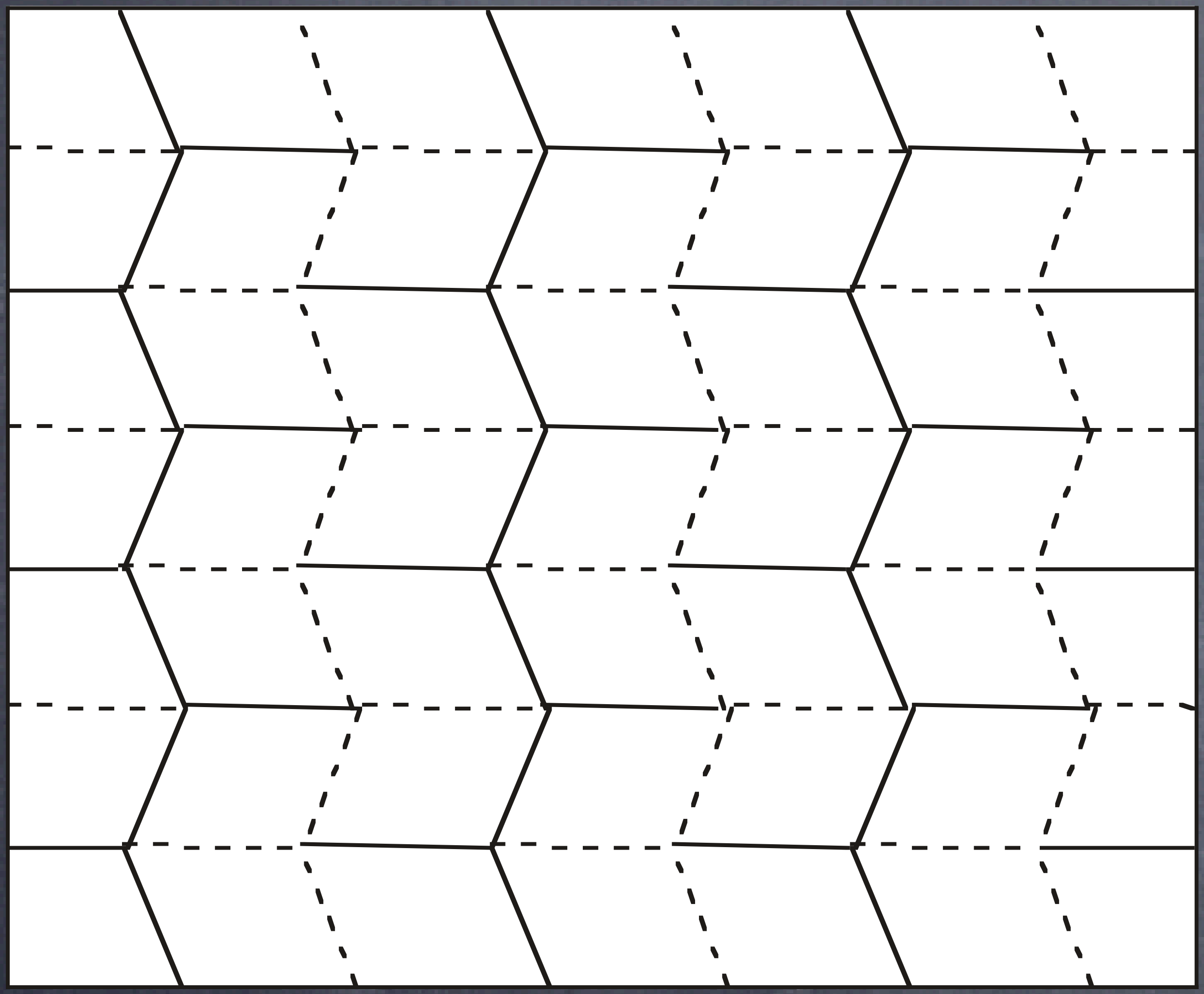


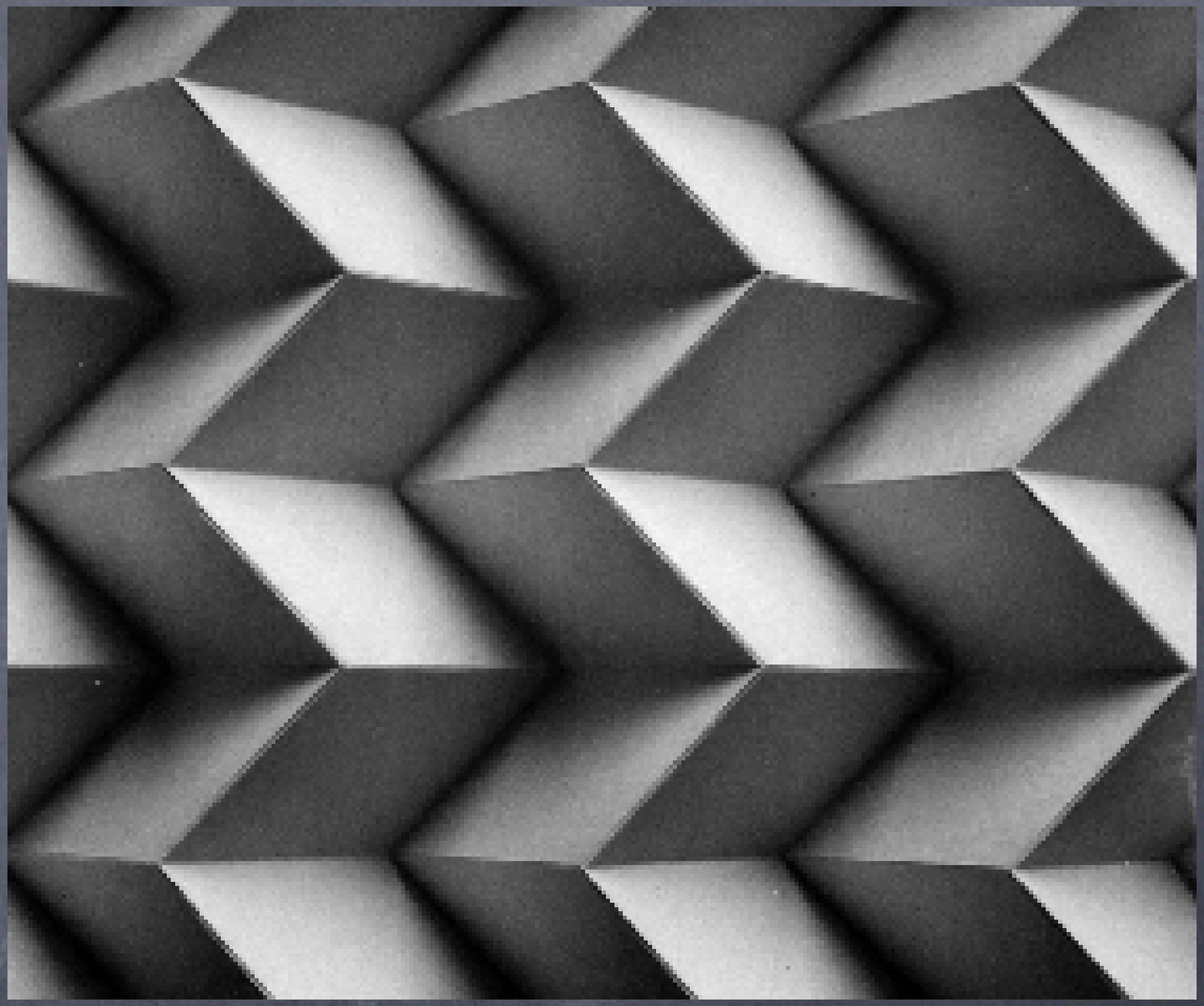


Hexagonal Wrap (Hull, 2005)

Applications

Miura's Map Fold





Applications

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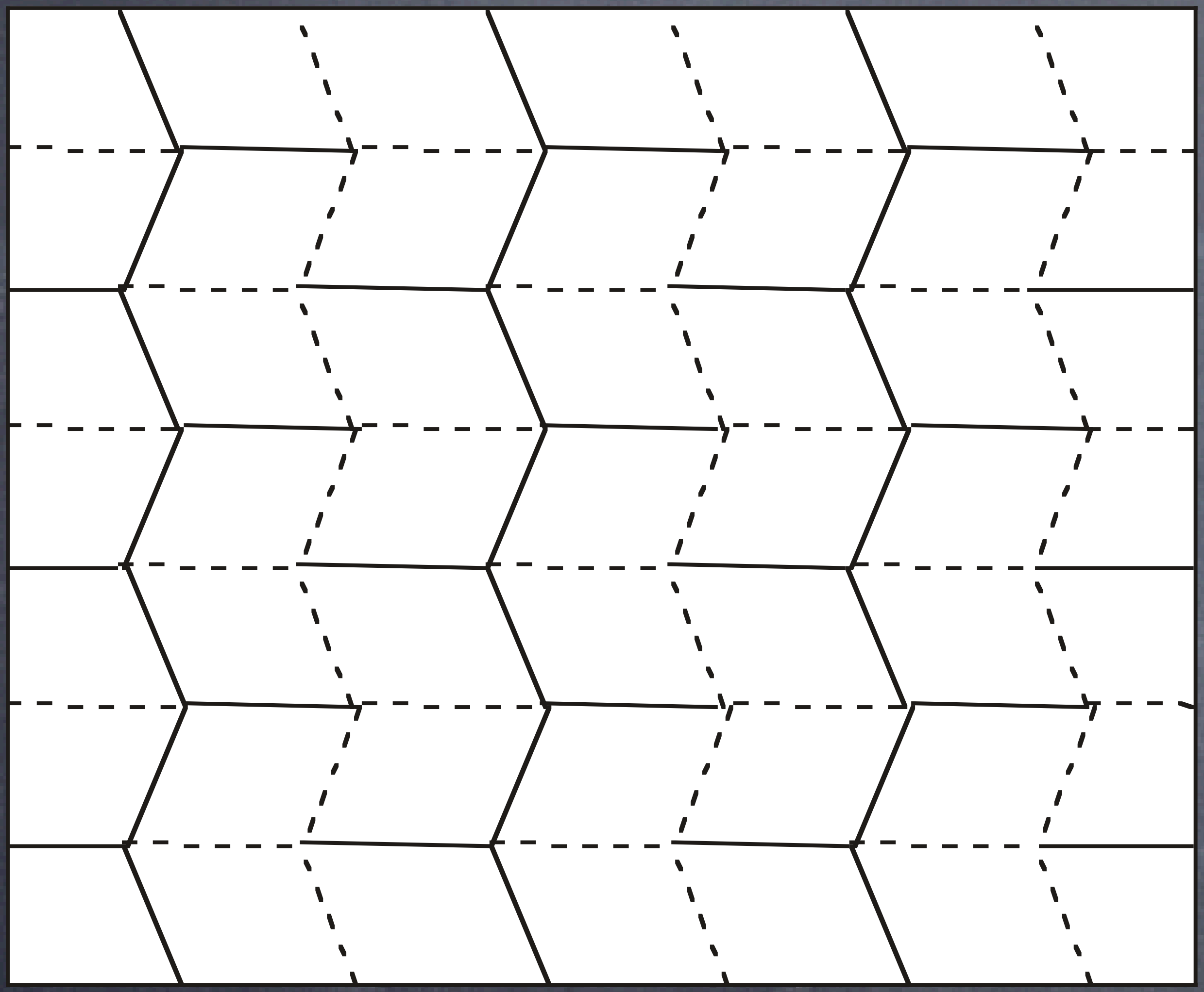

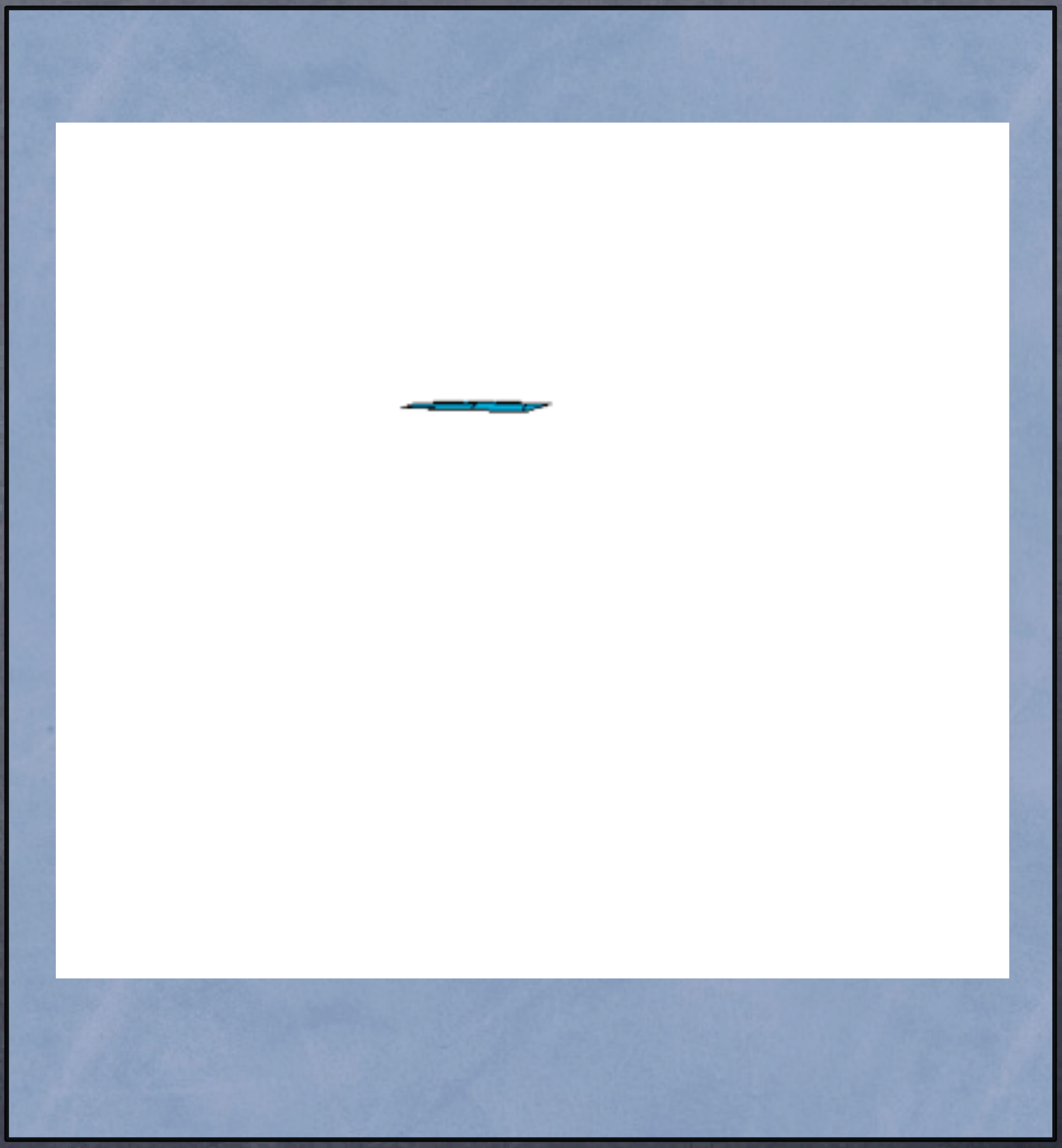

Applications

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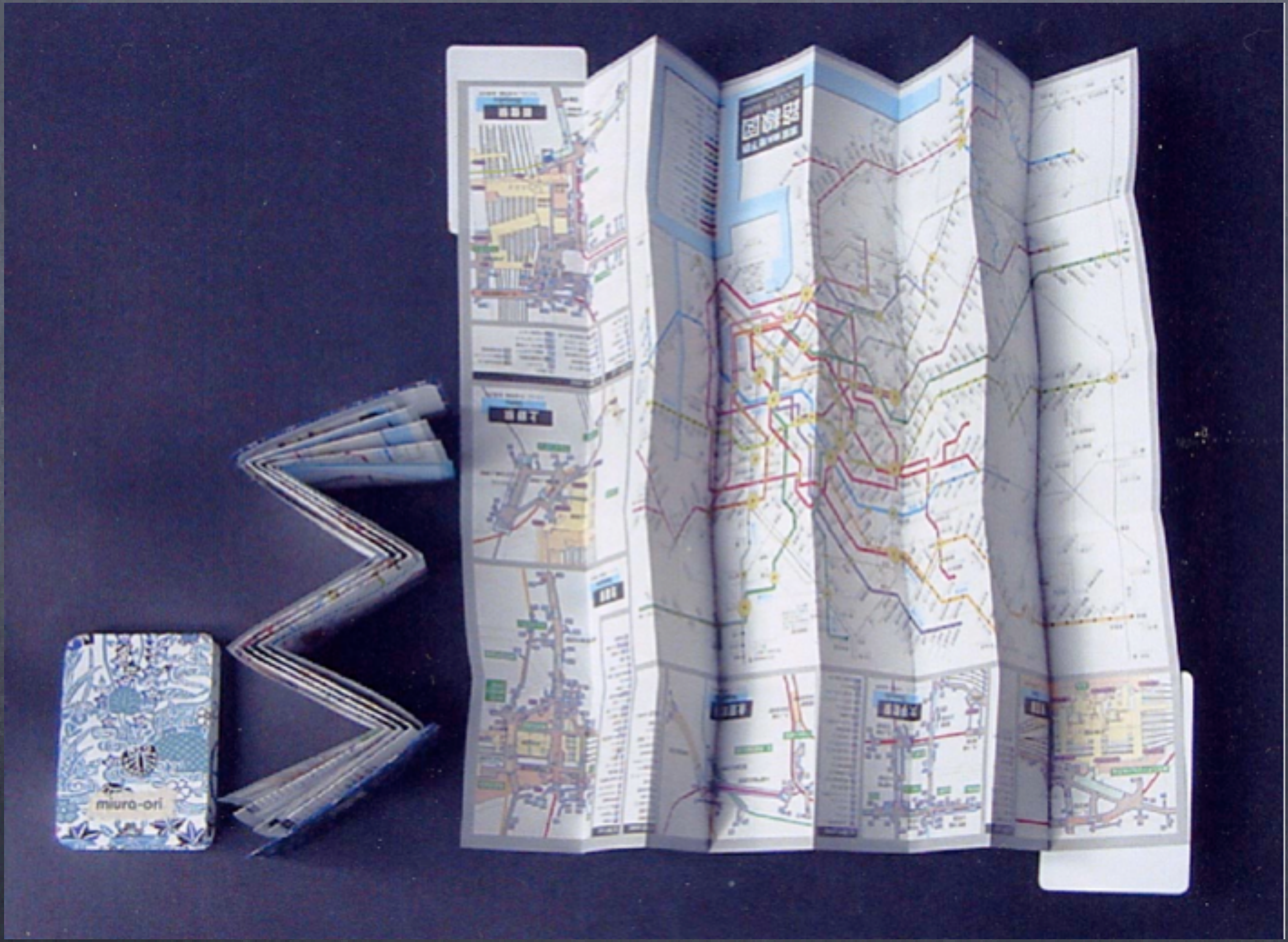
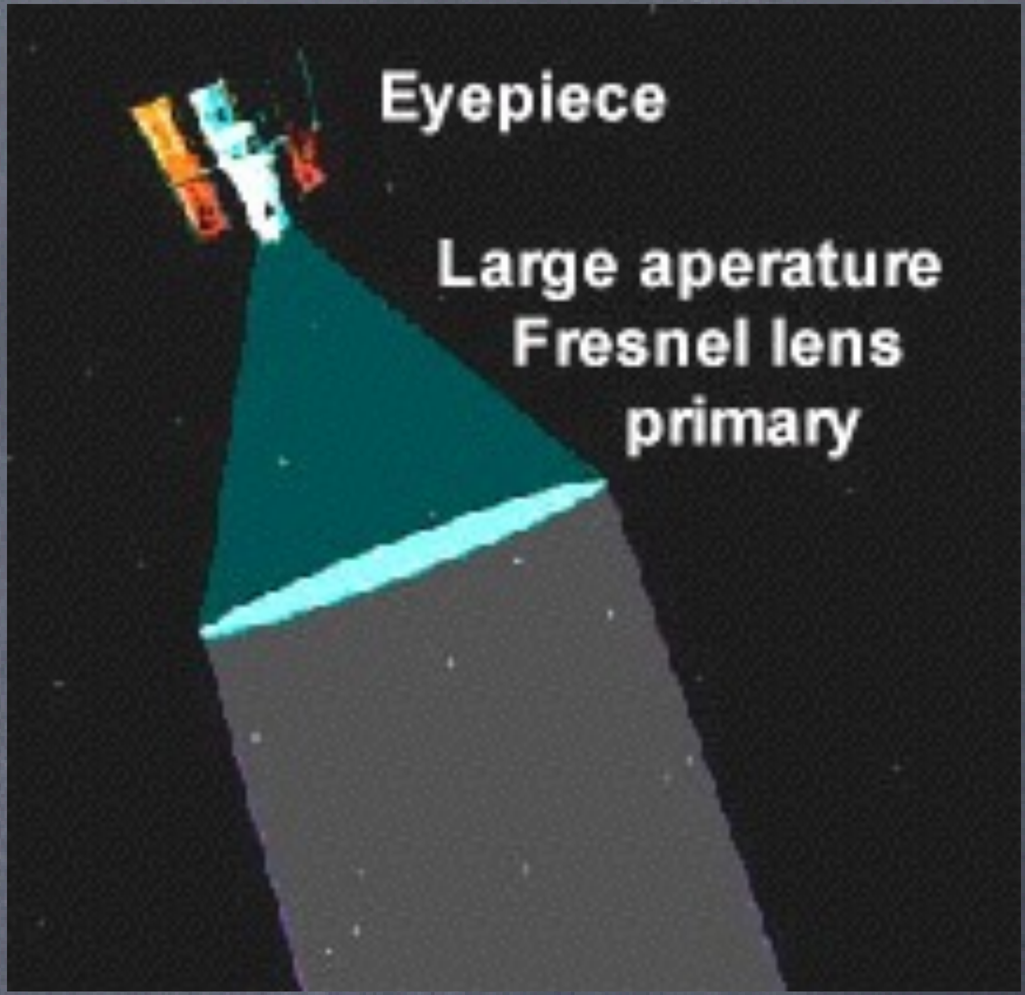


Applications

- Lawrence Livermore National Lab was thinking of making "better Hubble"



Applications

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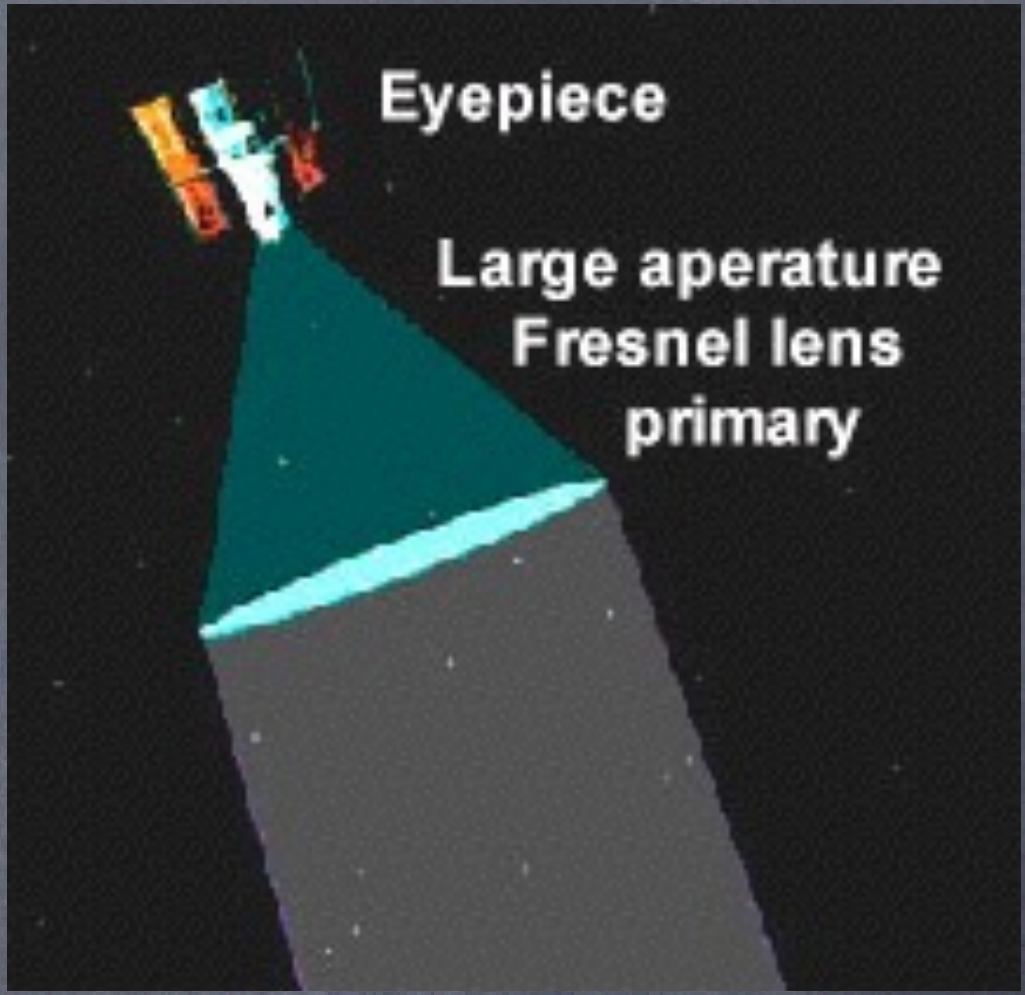
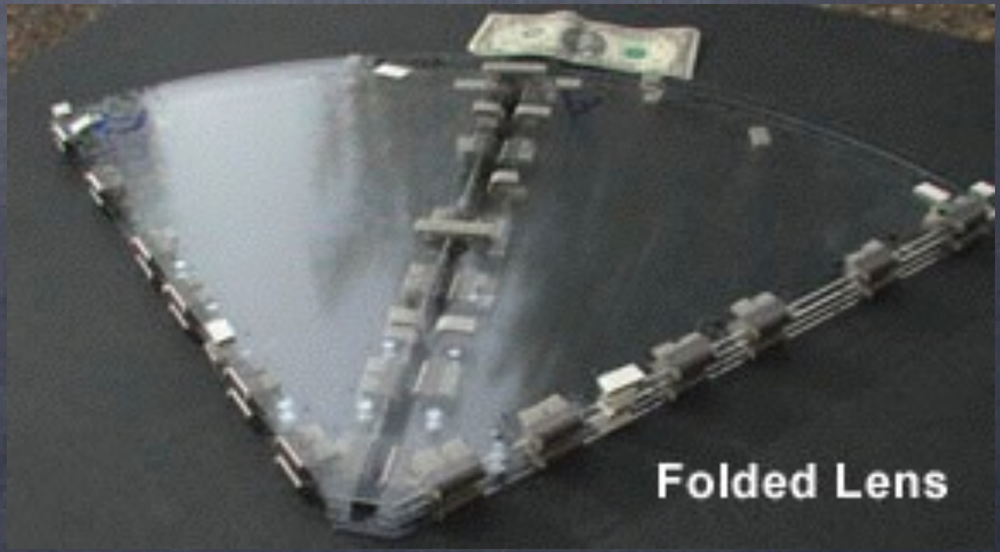
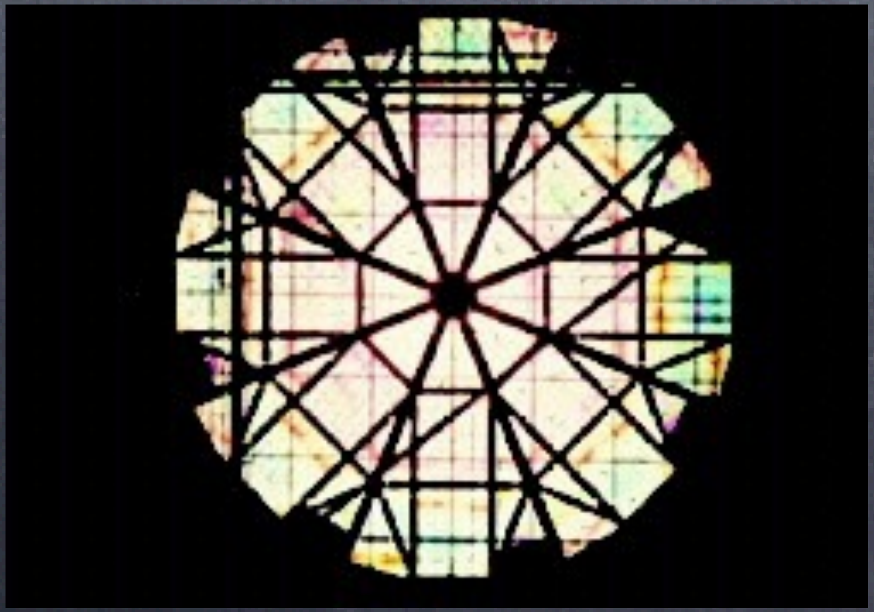
- To get it into space, it needs to "fold up" somehow.





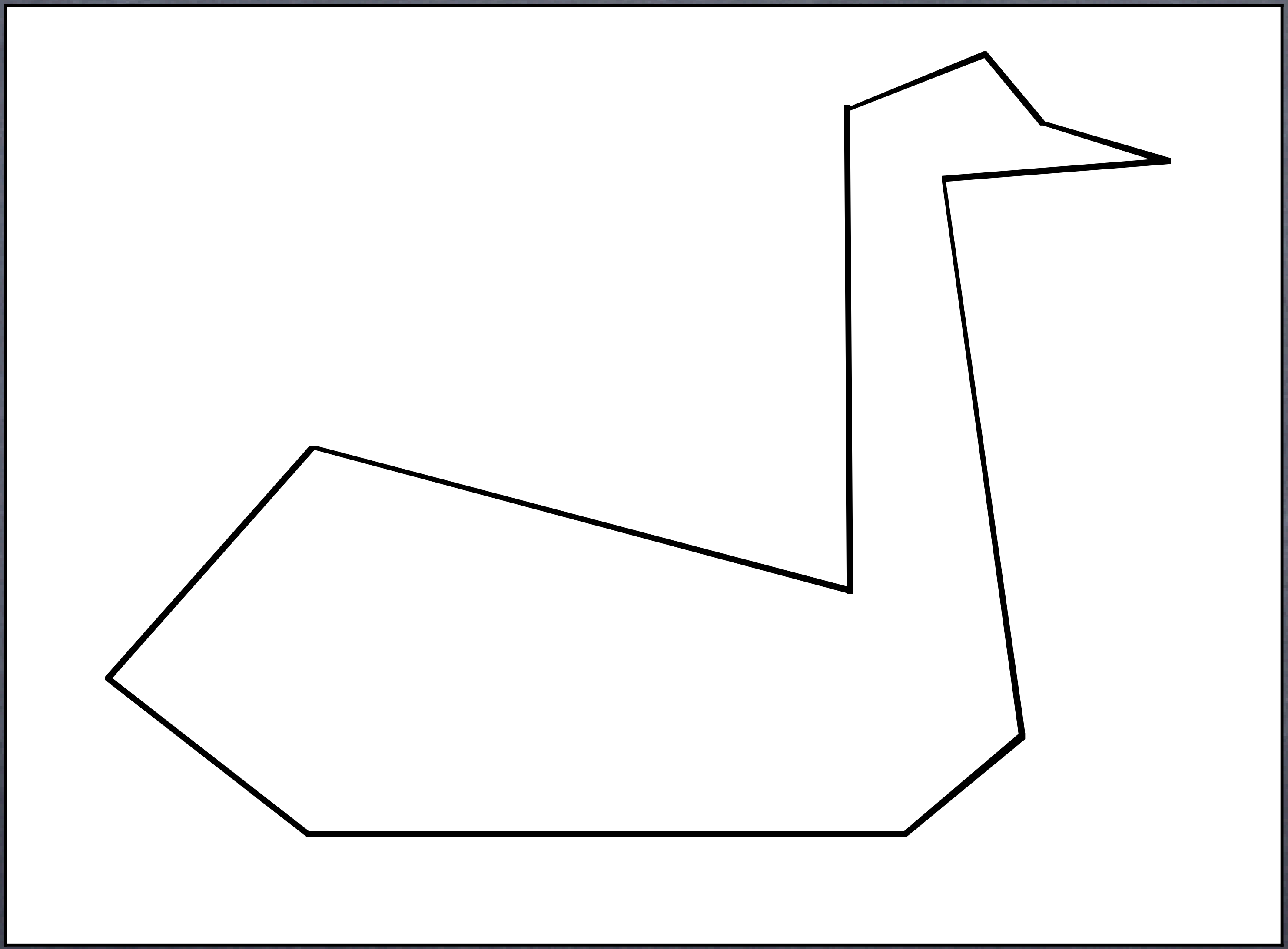
Applications

- Lawrence Livermore National Lab was thinking of making "better Hubble"
- To get it into space, it needs to "fold up" somehow.
- That's when they called Robert Lang.



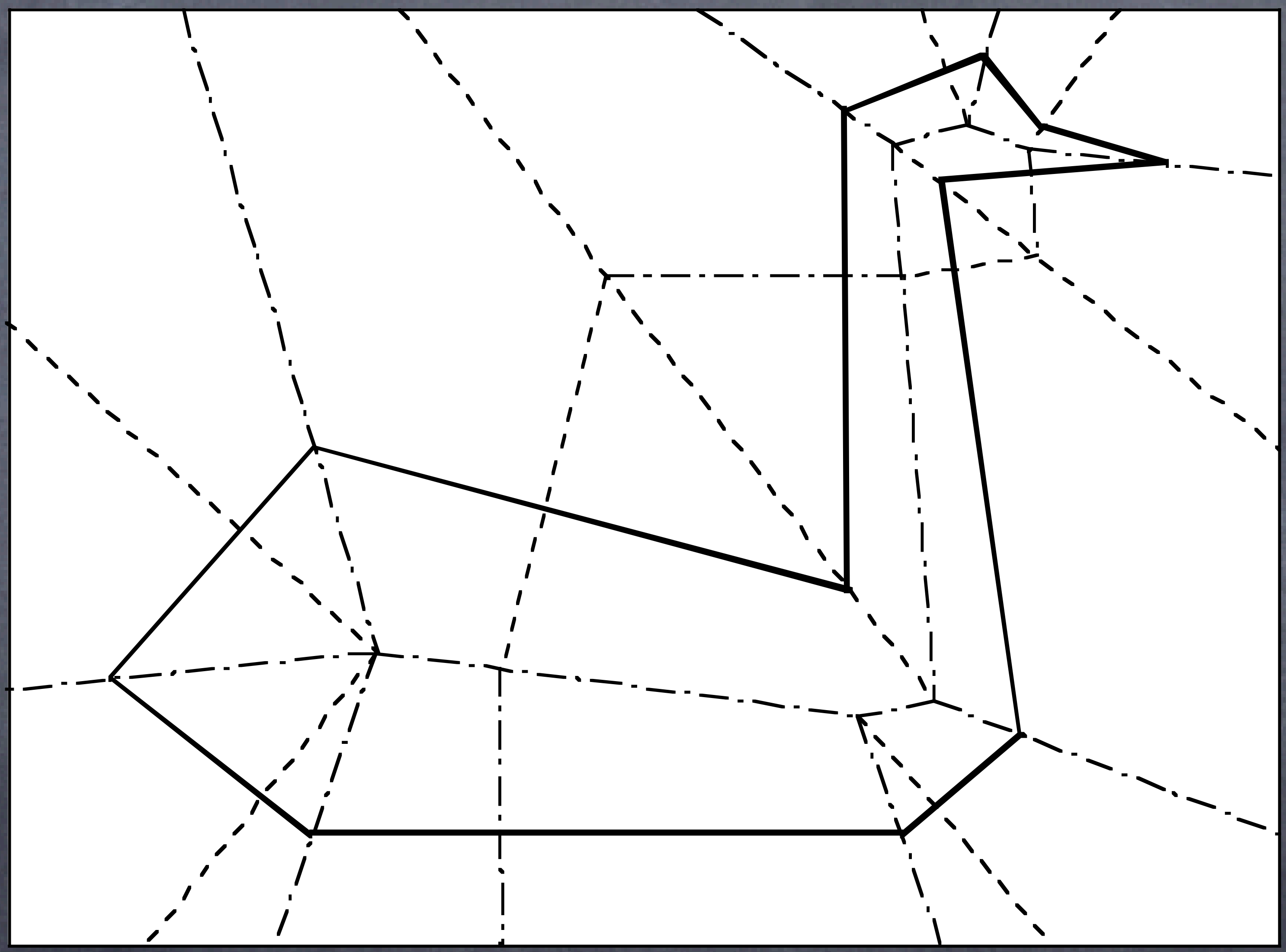
Applications

The Fold and Cut Problem



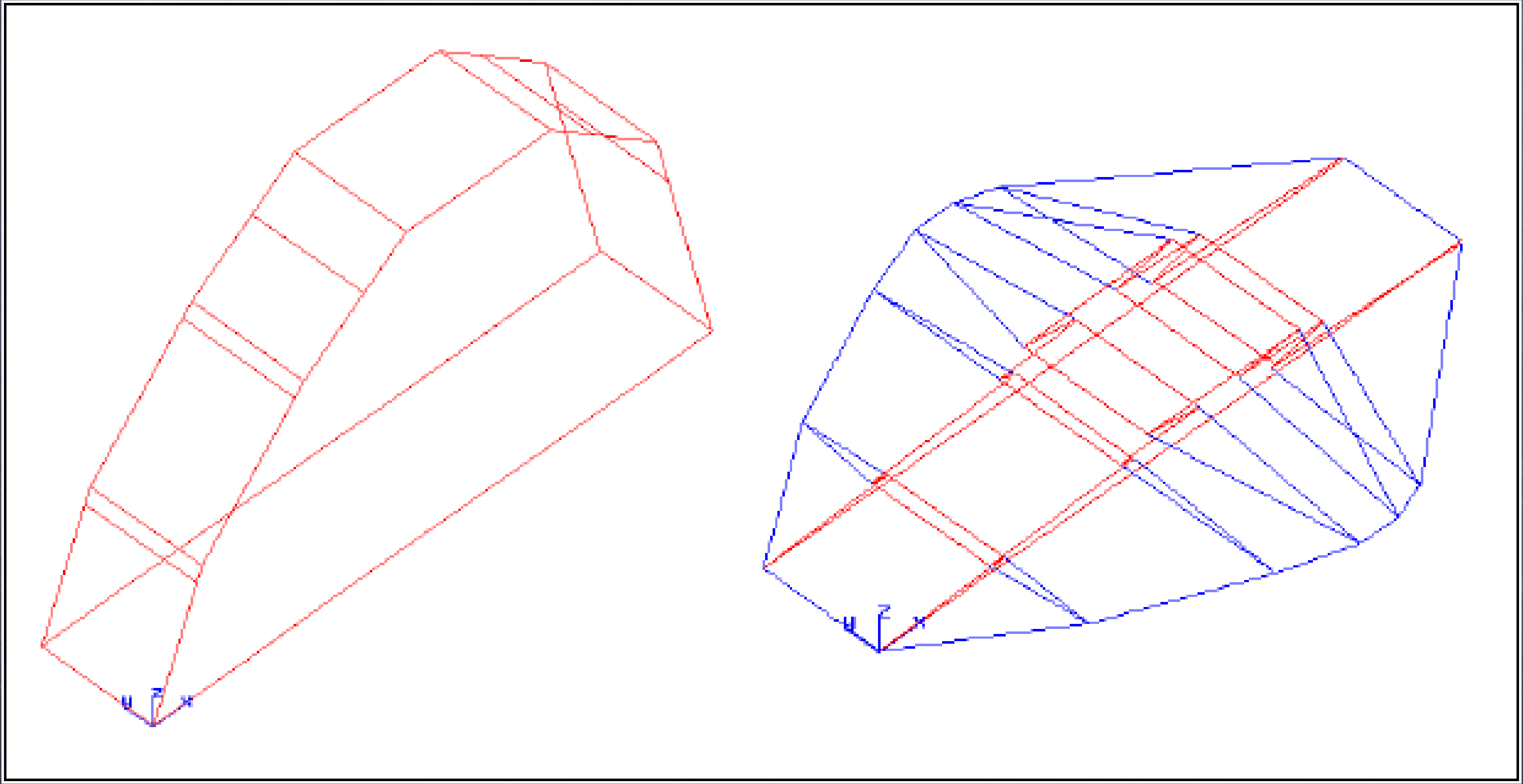
Applications

The Fold and Cut Problem



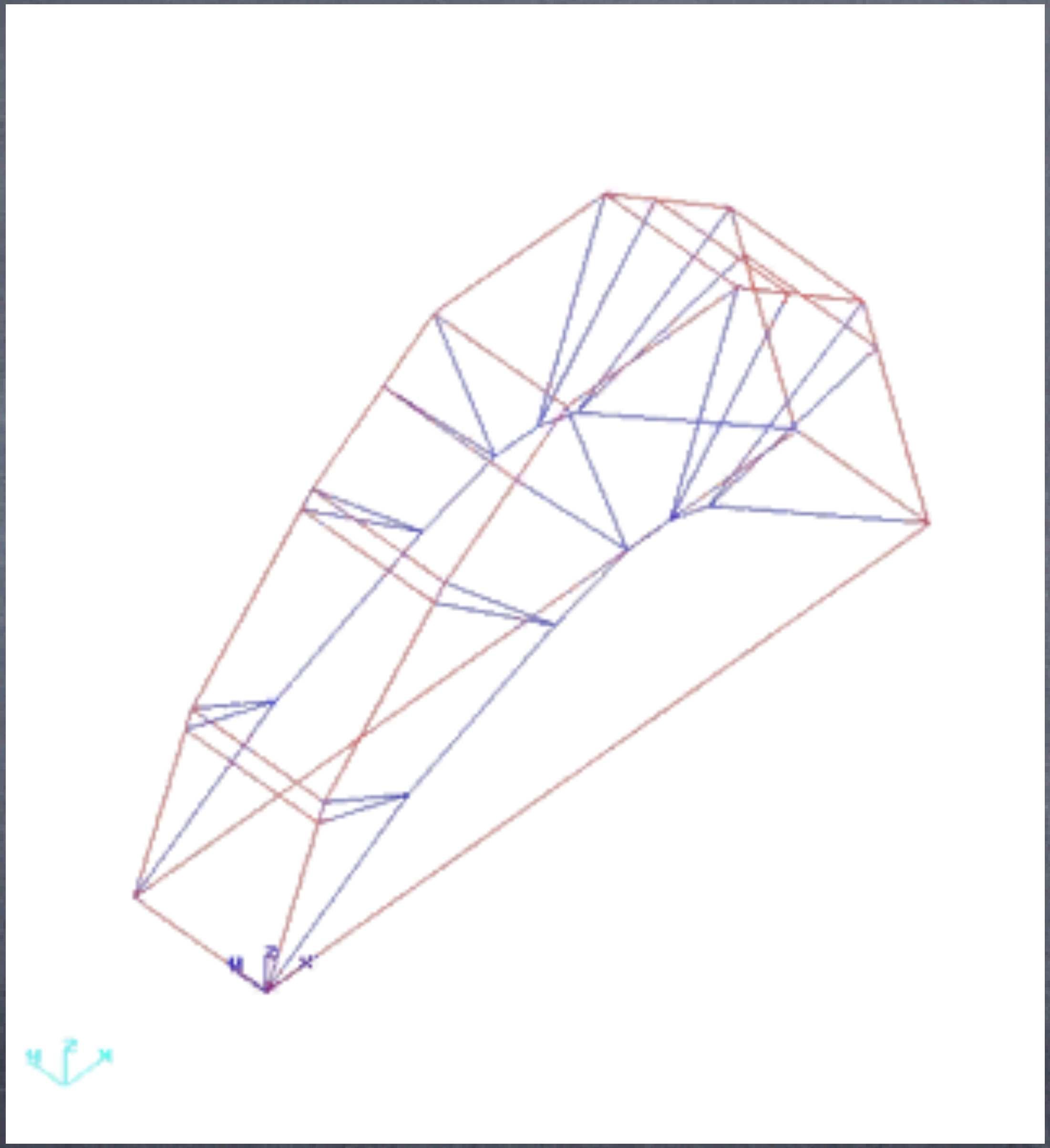
Demaine, Demaine, and Lubiw, 1999

Applications



Airbag Folding: EASi Engineering, Germany

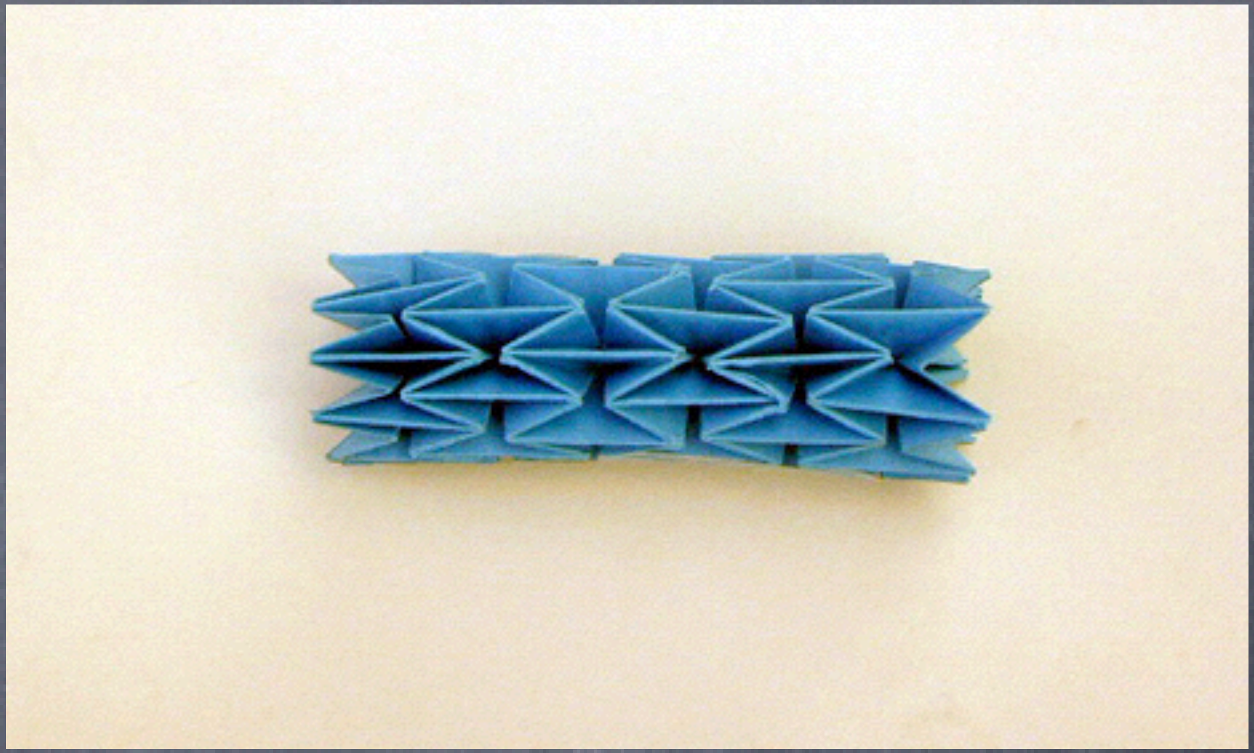
Applications

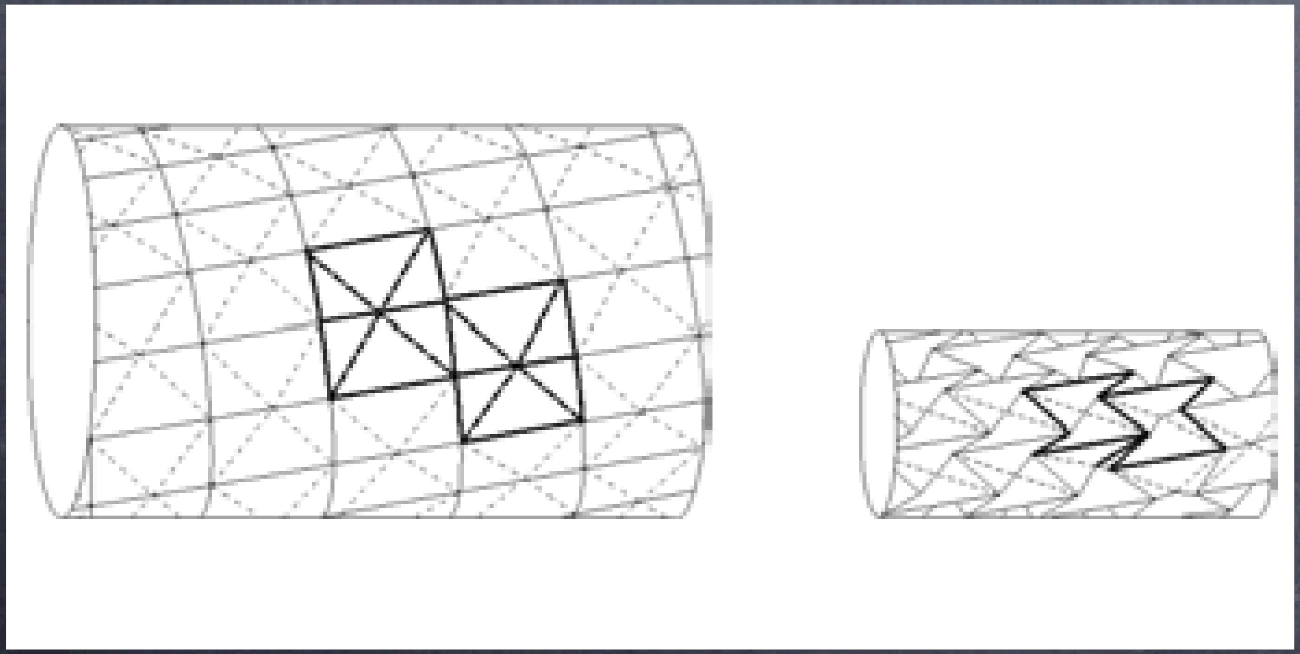


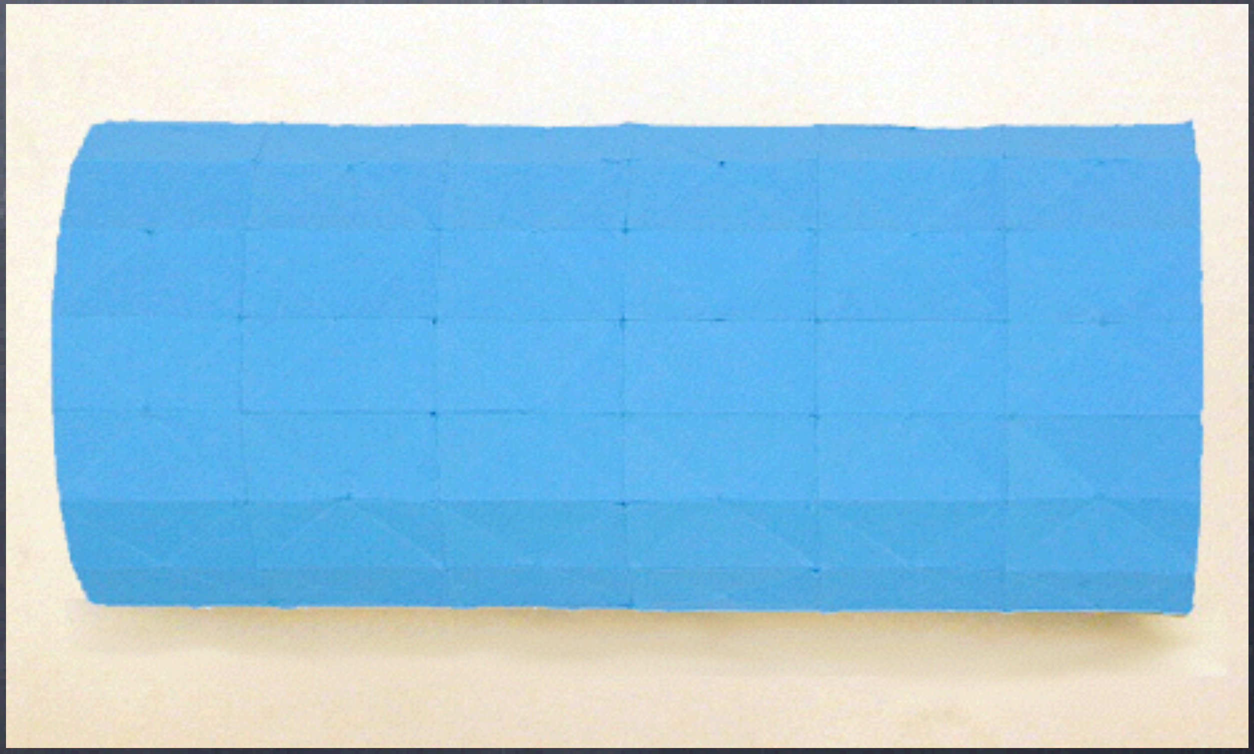
Airbag Folding: EASi Engineering, Germany

Applications

- An origami "stent" used to open up clogging arteries.







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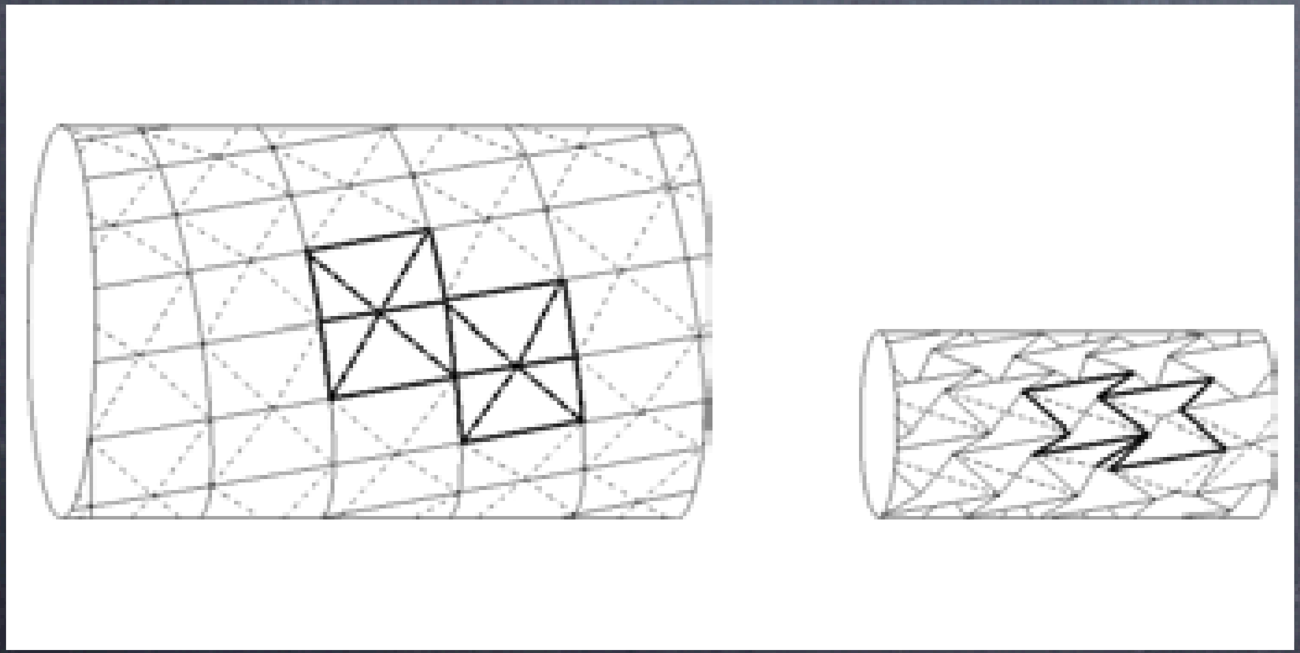
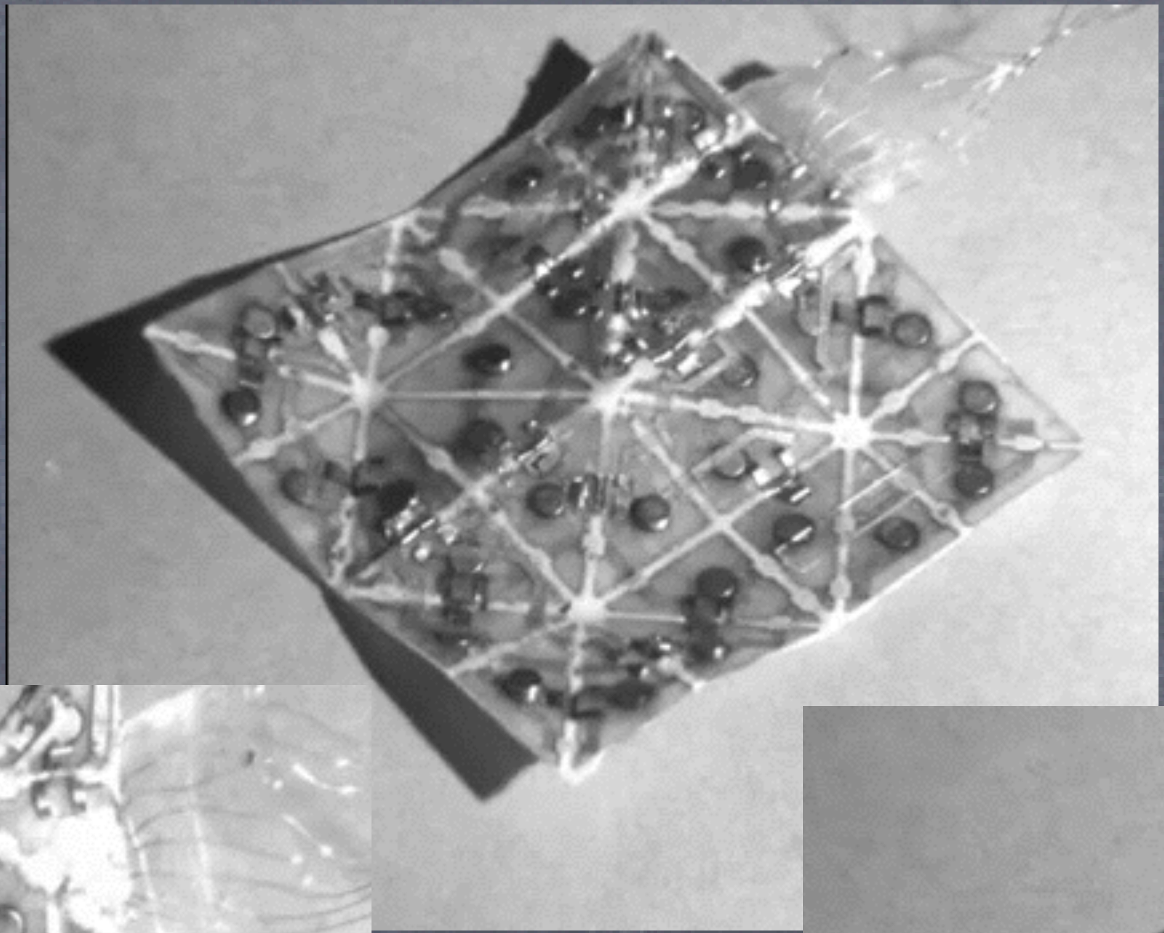
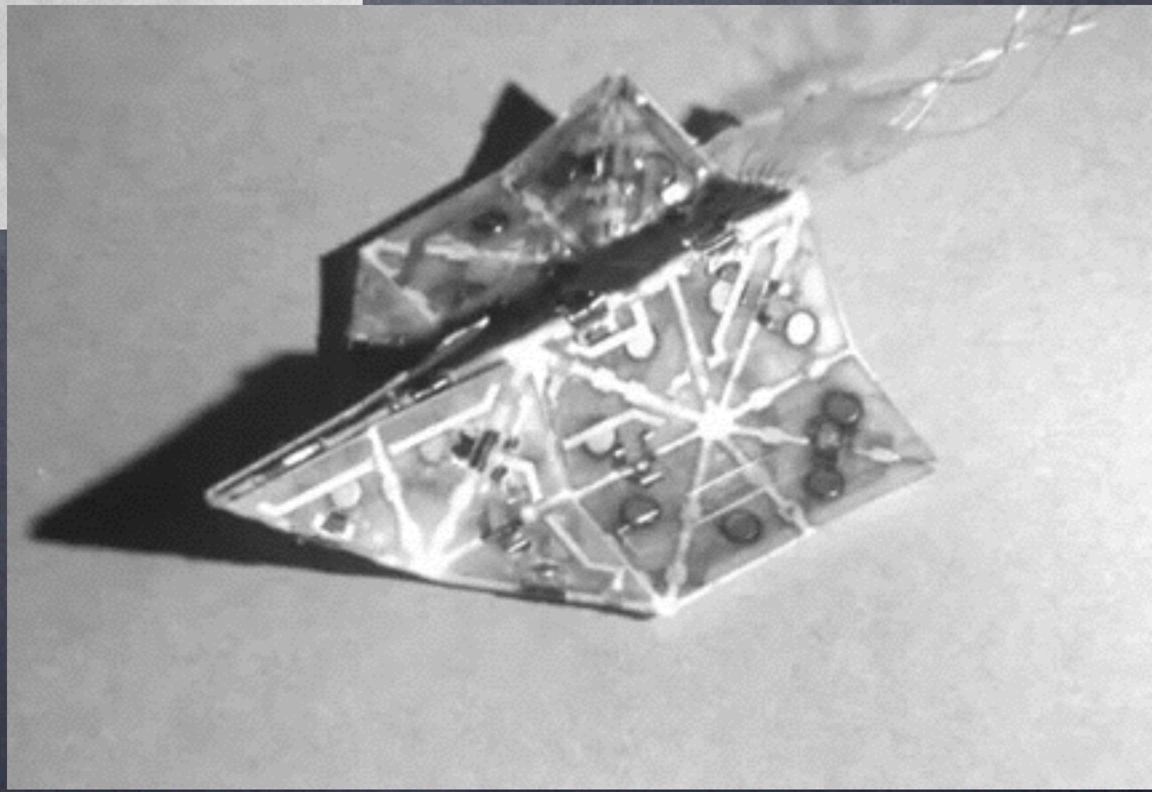


Applications

- Origami solutions to "Programmable Matter" : Harvard Microrobotics Lab





Applications

- The NSF and Air Force seem to care...

EMERGING FRONTIERS IN RESEARCH AND INNOVATION 2012 (EFRI-2012)

1. Flexible Bioelectronics Systems (BioFlex)
2. Origami Design for Integration of Self-assembling Systems for Engineering Innovation (ODISSEI)
3. Photosynthetic Biorefineries (PSBR)

PROGRAM SOLICITATION
NSF 11-571

REPLACES DOCUMENT(S):
NSF 10-596

National Science Foundation

Directorate for Engineering
Emerging Frontiers in Research and Innovation

Directorate for Mathematical & Physical Sciences
Division of Mathematical Sciences
Division of Materials Research

Directorate for Biological Sciences
Division of Molecular and Cellular Biosciences

Air Force Office of Scientific Research

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Air Force Office of Scientific Research

transformative BioFlex research would have the long-term potential to enhance the quality of patient care, while reducing the total cost of healthcare delivery.

2) Origami Design for Integration of Self-assembling Systems for Engineering Innovation (ODISSEI)

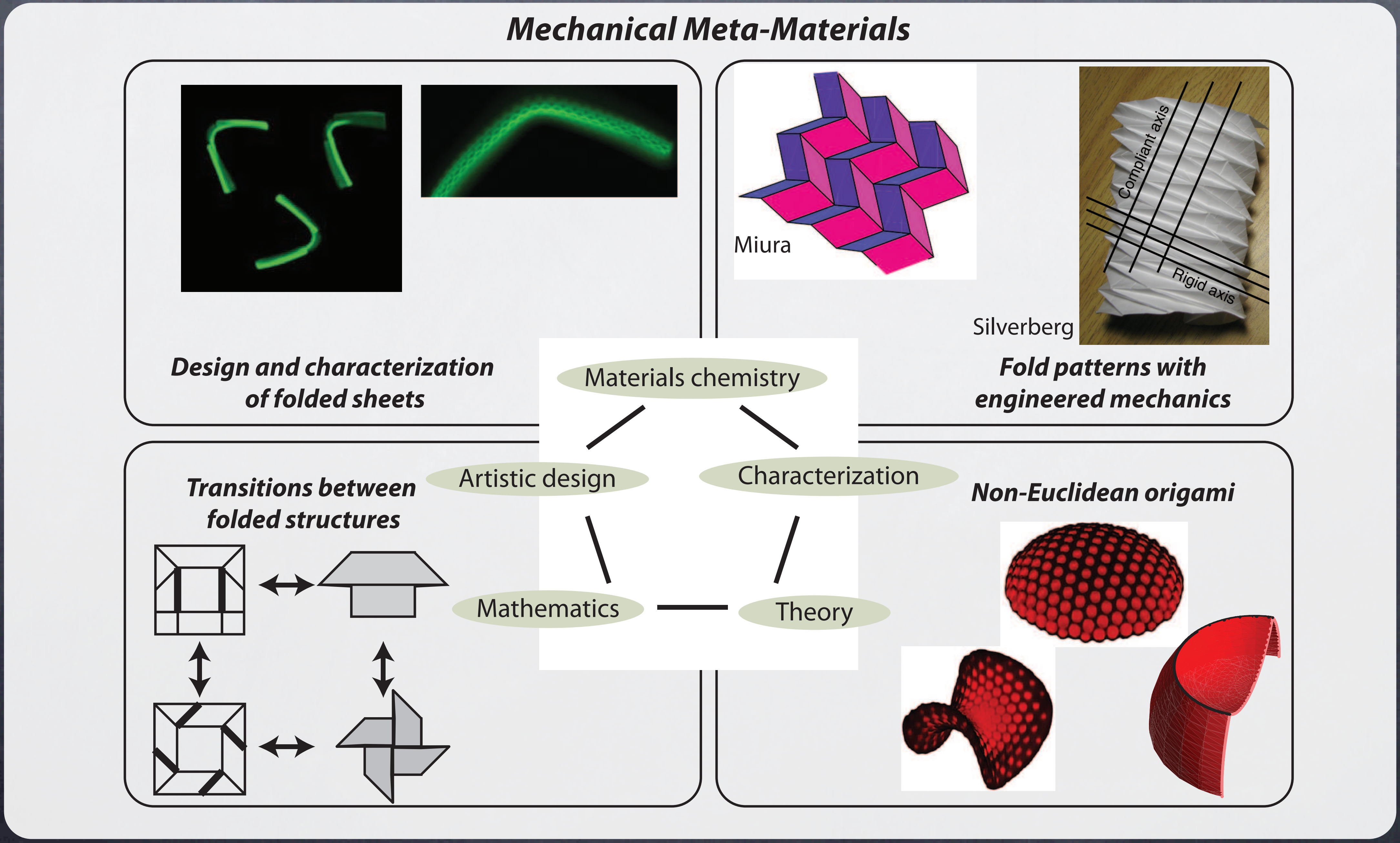
The central theme of the ODISSEI initiative is to explore the folding and unfolding of materials and structures at all scales and across scales in order to overcome obstacles associated with the rigorous design of engineered systems. Just as Homer's epic poem, *Odyssey*, relates Odysseus' long journey home following the fall of Troy, ODISSEI represents a journey into an emerging field of science and engineering that encourages the exploration of origami engineering for the design of self-assembling, multifunctional, compliant structures facilitated through the integration of active materials, design theory, mathematics (geometric origami), and artistic expression. A fundamental understanding of how materials at different scales can be designed to be folded and unfolded, and the relation of such folding to self-assembly, will have tremendous impact in numerous areas of national priority. Such engineered systems would have application in the design of materials, compliant mechanisms, and structures, impacting critical industries including manufacturing, energy, and biomedicine.

Traditionally, origami has been recognized as the art of paper folding. In recent decades, however, engineers, scientists, and mathematicians have used principles of folding in materials design to achieve substantially more complex shapes and products for applications spanning scales and disciplines, including automotive safety (e.g. airbags), biomedical device design (e.g. deployable heart stents), and DNA Origami for patterning. Origami structures can also be found in nature, such as in the growth process of plant leaves, providing possibilities for biomimicry. Present barriers to folding and unfolding mechanisms pertain to lack of understanding in scaling laws, the inability to easily reorient matter to achieve the desired folding and unfolding, the lack of understanding pertaining to smaller scale influence on larger scale compliance, and the lack of appropriate mathematics and tools to enable rigorous design of foldable engineered systems. Fundamental theories associated with folding and unfolding (in the sciences as well as mathematics) would enhance the ability to achieve rigorous designs for engineered systems and would have the potential to significantly advance numerous areas of national priority.

3) Photosynthetic Biorefineries (PSBR)

Achieving the sustainable production of transportation fuels and industrial chemicals will be one of the grand challenges of the 21st century. One direct route to achieve this end is to harness solar energy to

- NSF EFRI-1240441: **Mechanical Meta-Materials from Self-Folding Polymer Sheets**
PI: Chris Santangelo (UMass Amherst)
co-PIs: Itai Cohen (Cornell), Ryan Hayward (UMass Amherst), and Tom Hull (me).

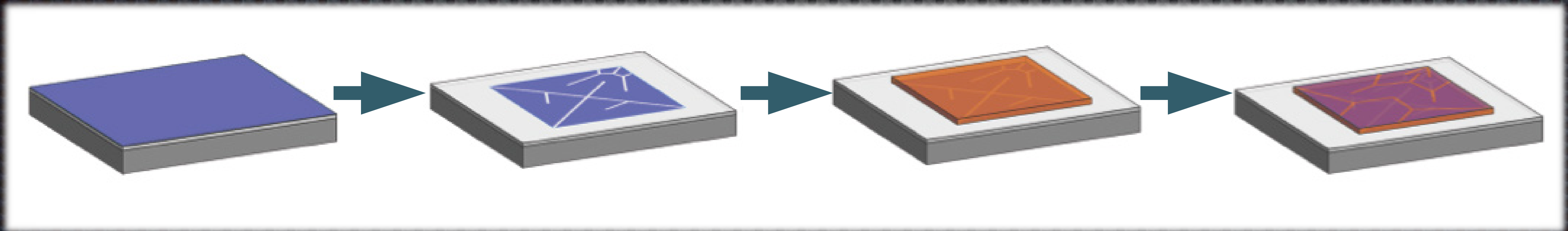


What are we doing with all this?

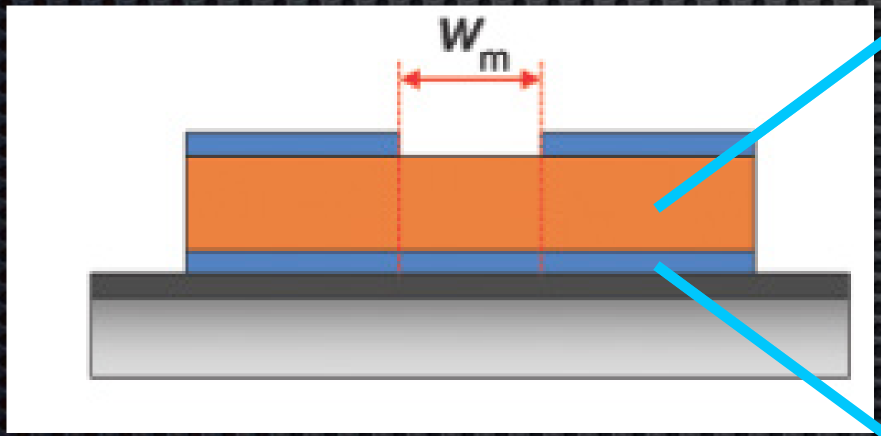
Ryan Hayward, Chris Santangelo, UMass
Itai Cohen, Cornell

Self-folding polymer gels

Photo-lithographic patterning of self-folding gel origami:

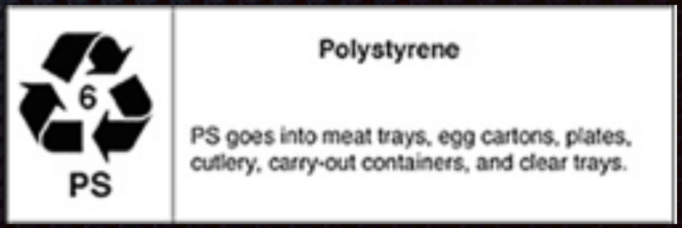


Schematic side-view of a mountain fold:



Temperature-responsive soft hydrogel layer
thickness: ~ 1 um
(100x thinner than paper)

Rigid plastic layer
(similar to polystyrene)
thickness: 70 nm



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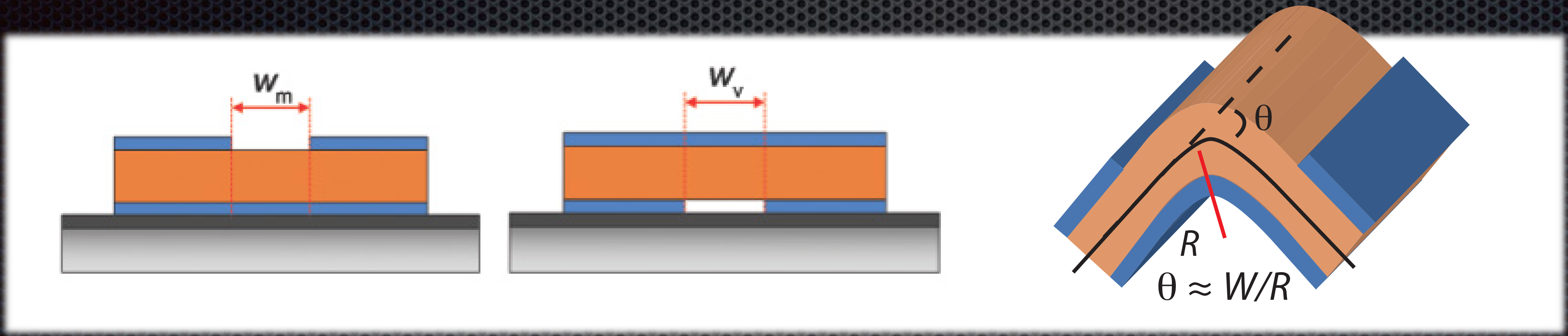
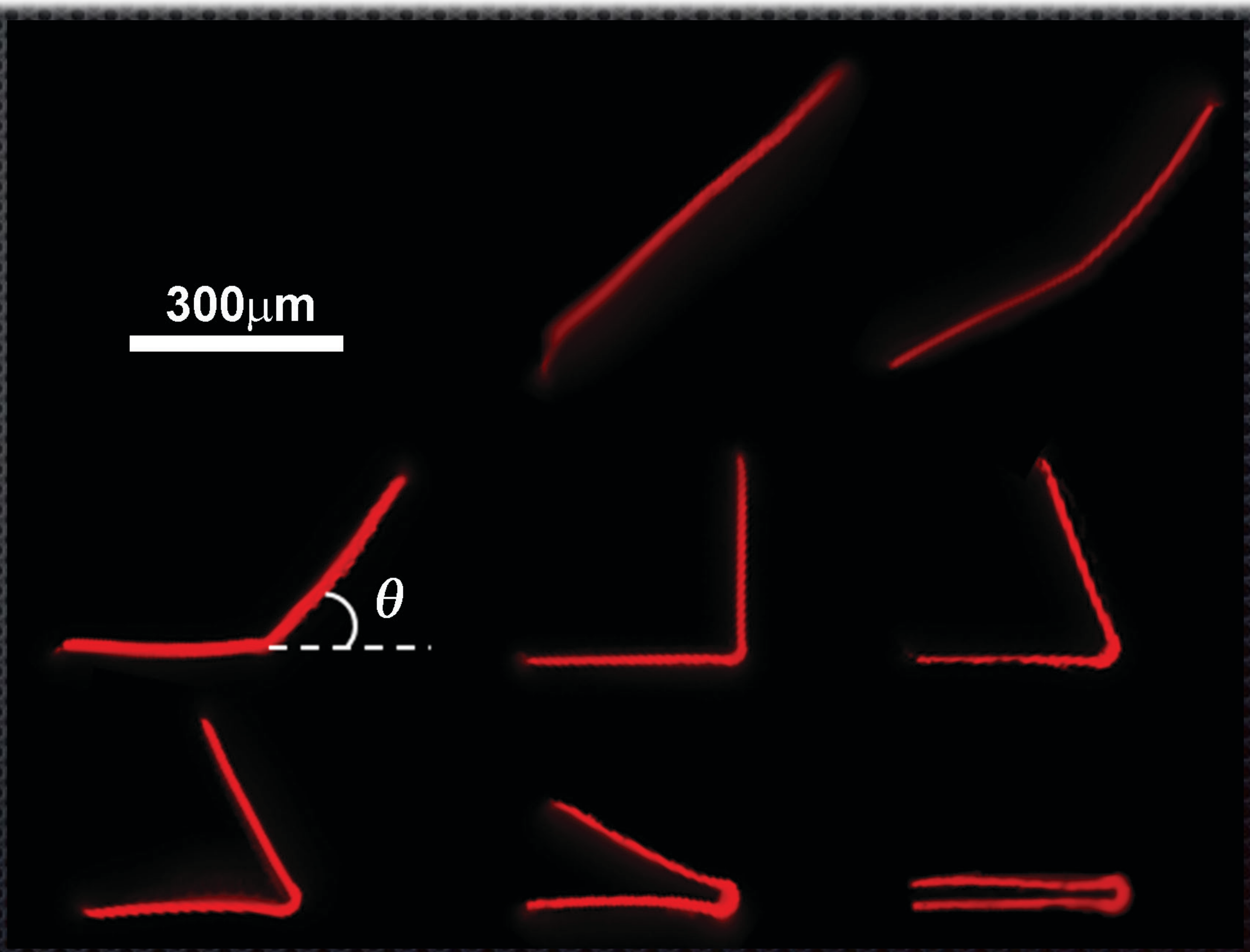
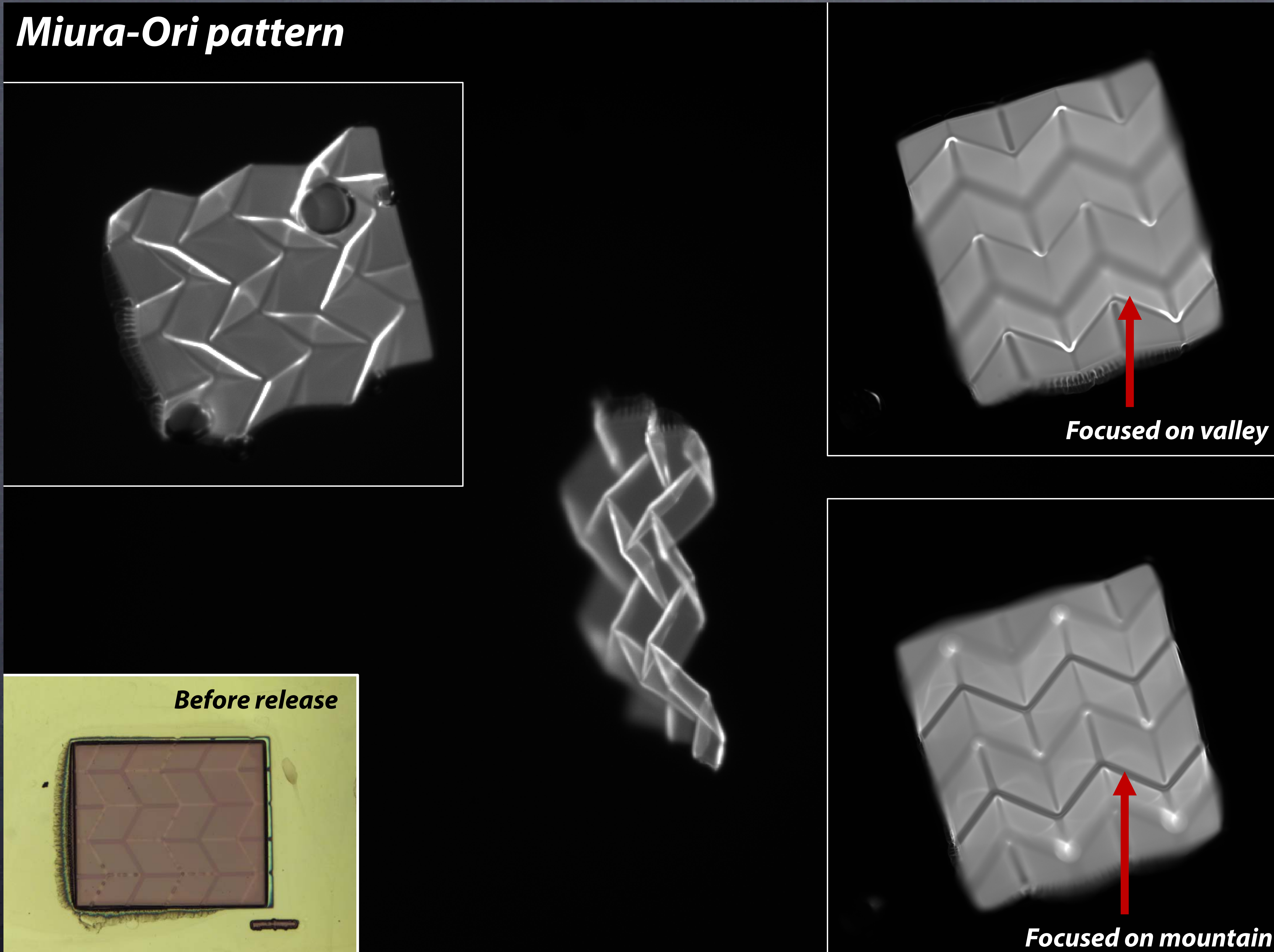
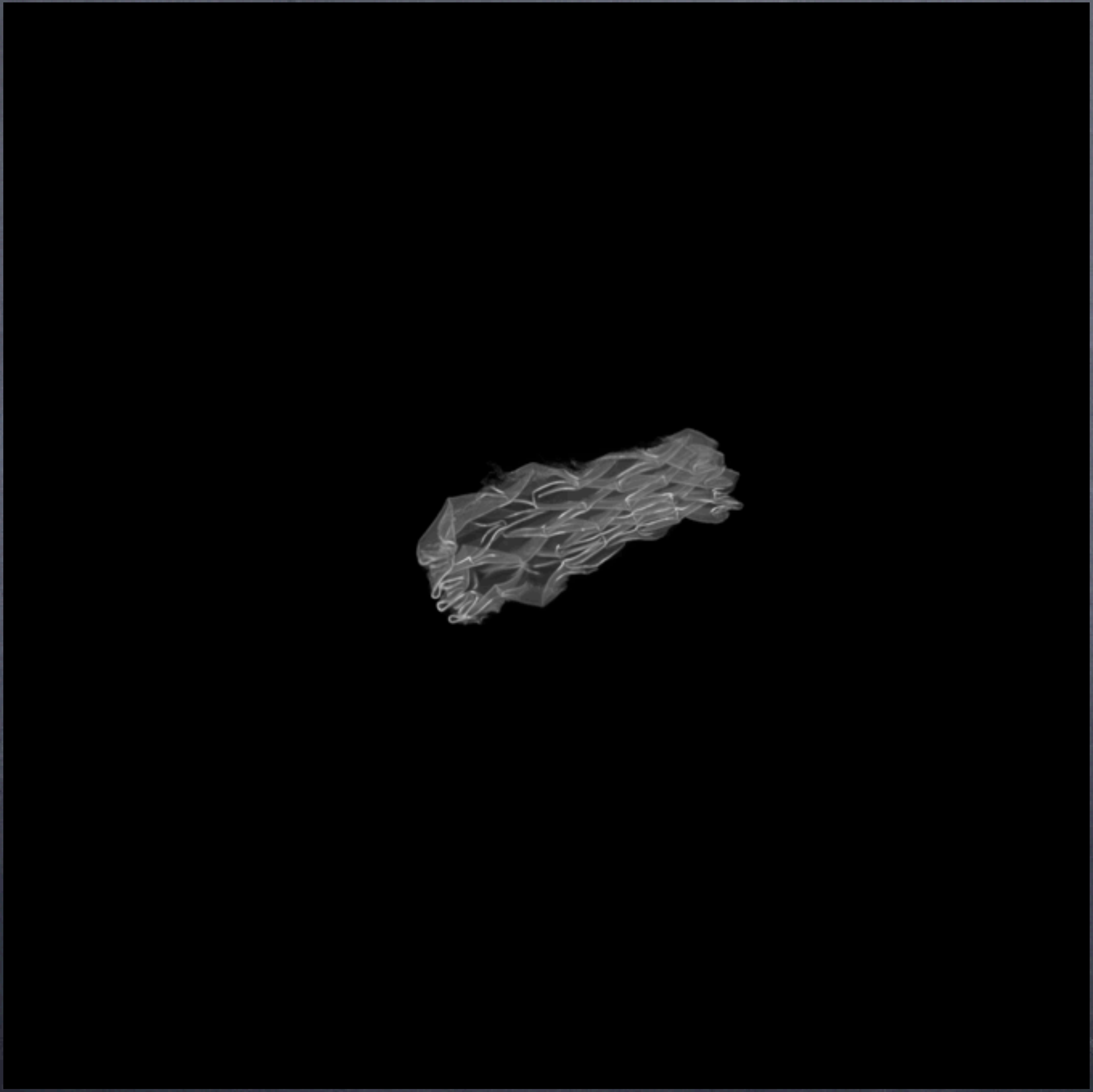




Miura-Ori pattern



Deswelling and reswelling



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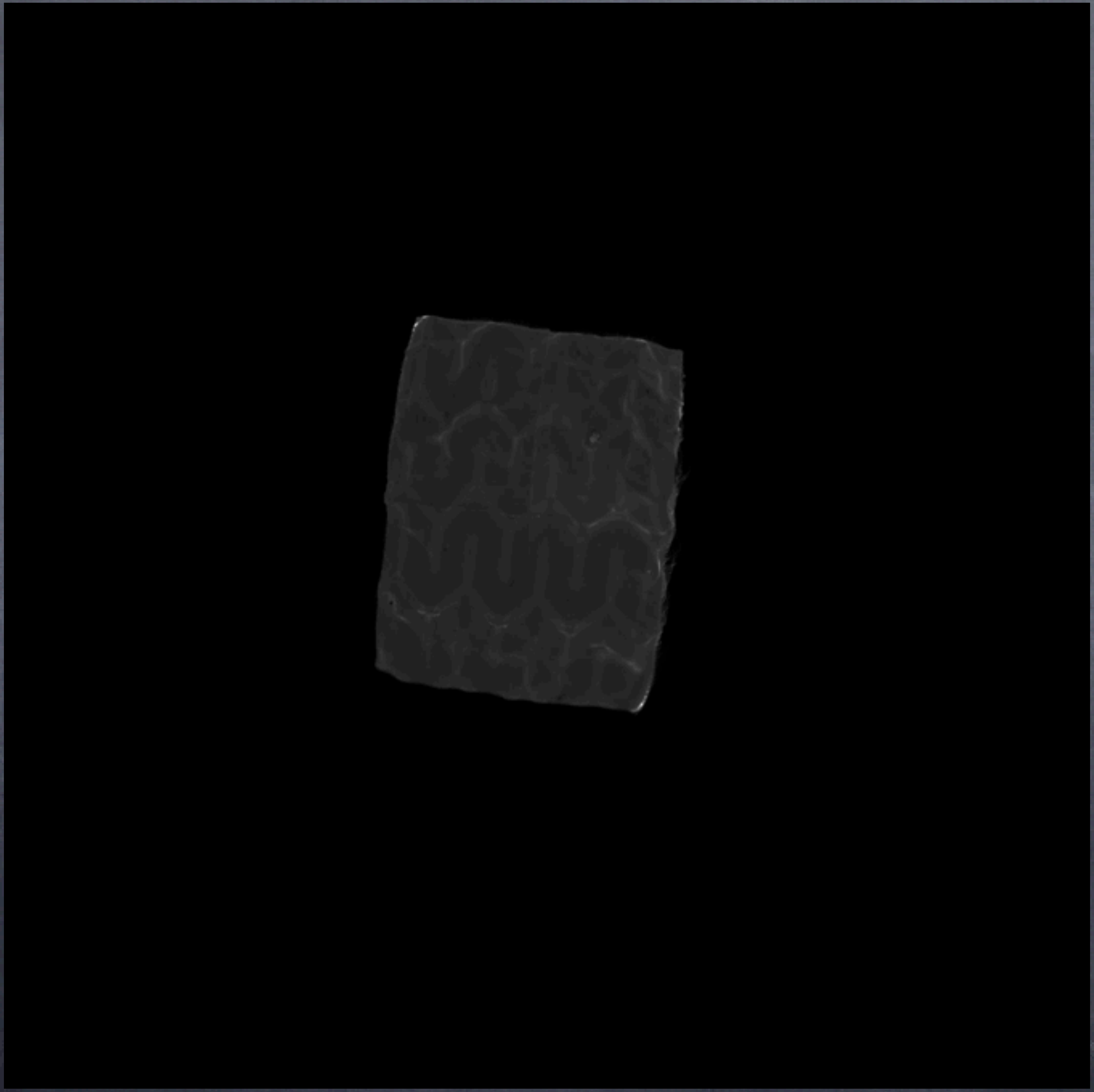
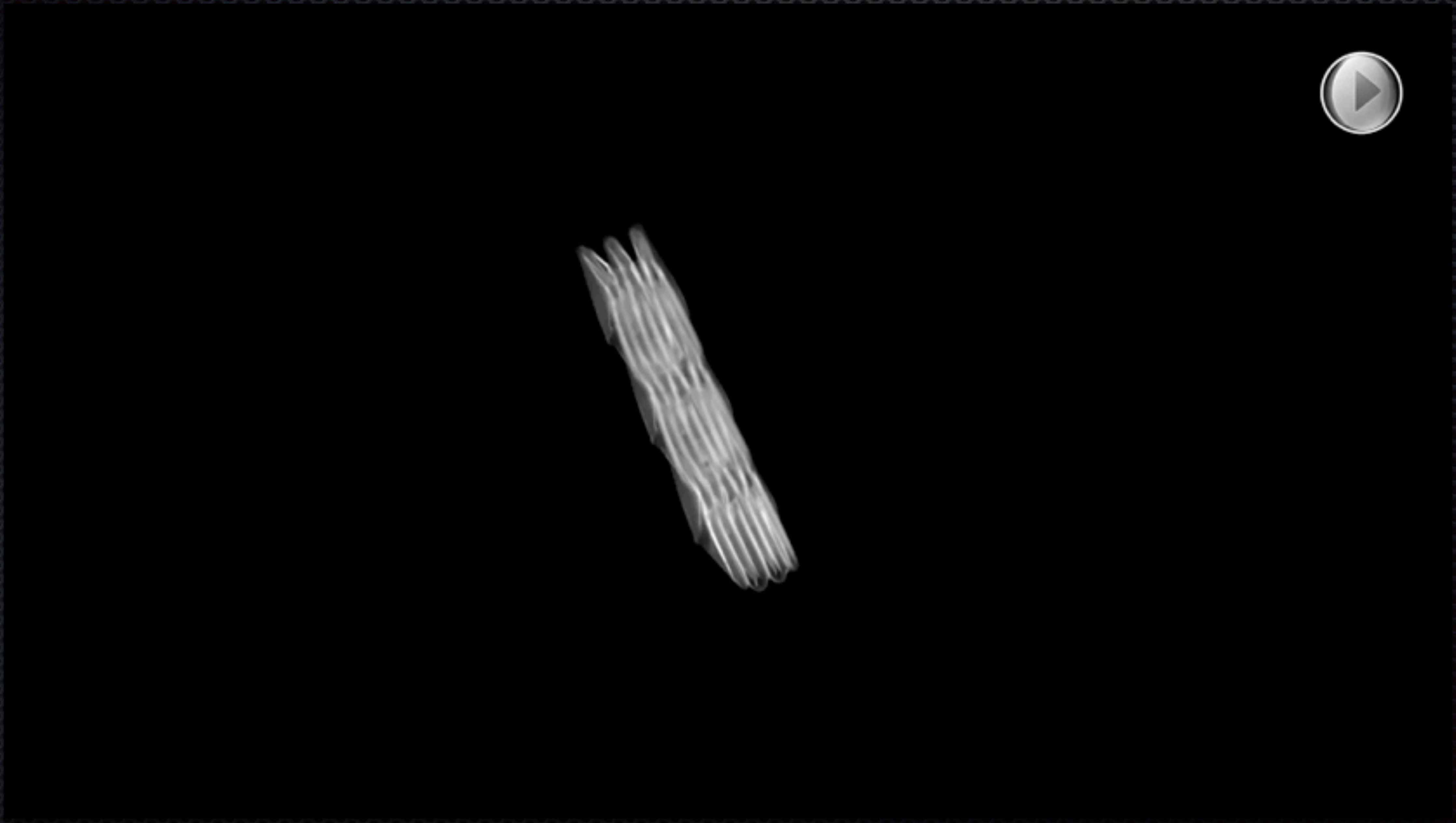
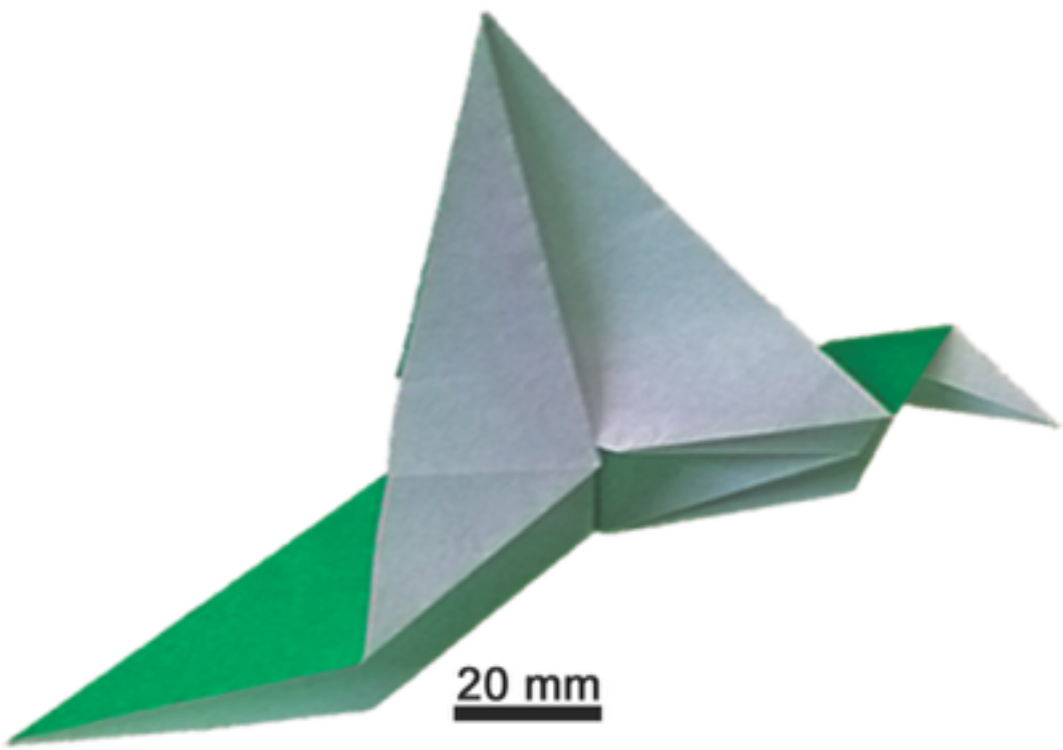


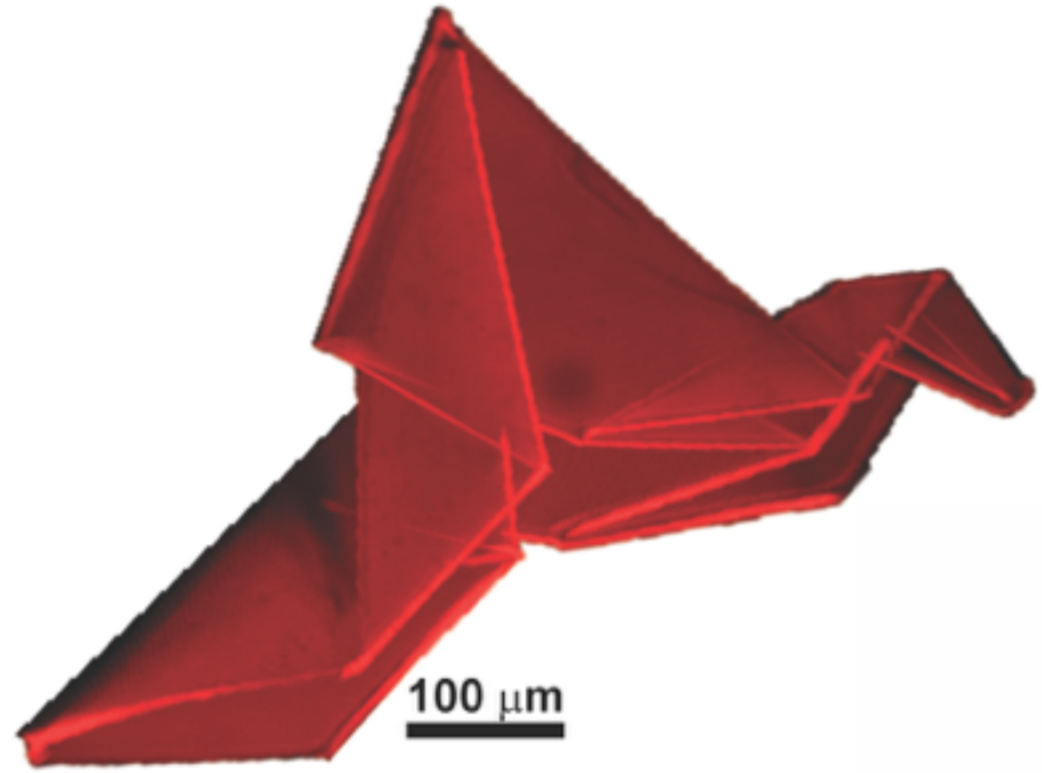
Self-folding Miura-ori

Ryan Hayward, Chris Santangelo, UMass
Itai Cohen, Cornell

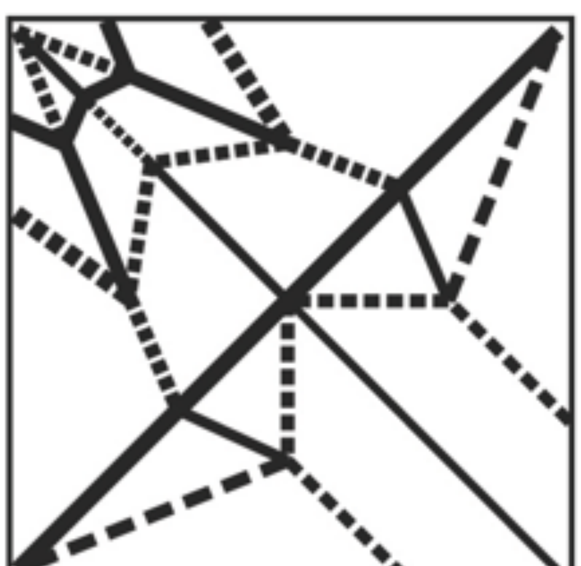


Can make (complex and tiny) origami with simultaneous folds





Randlett, “New Flapping Bird”



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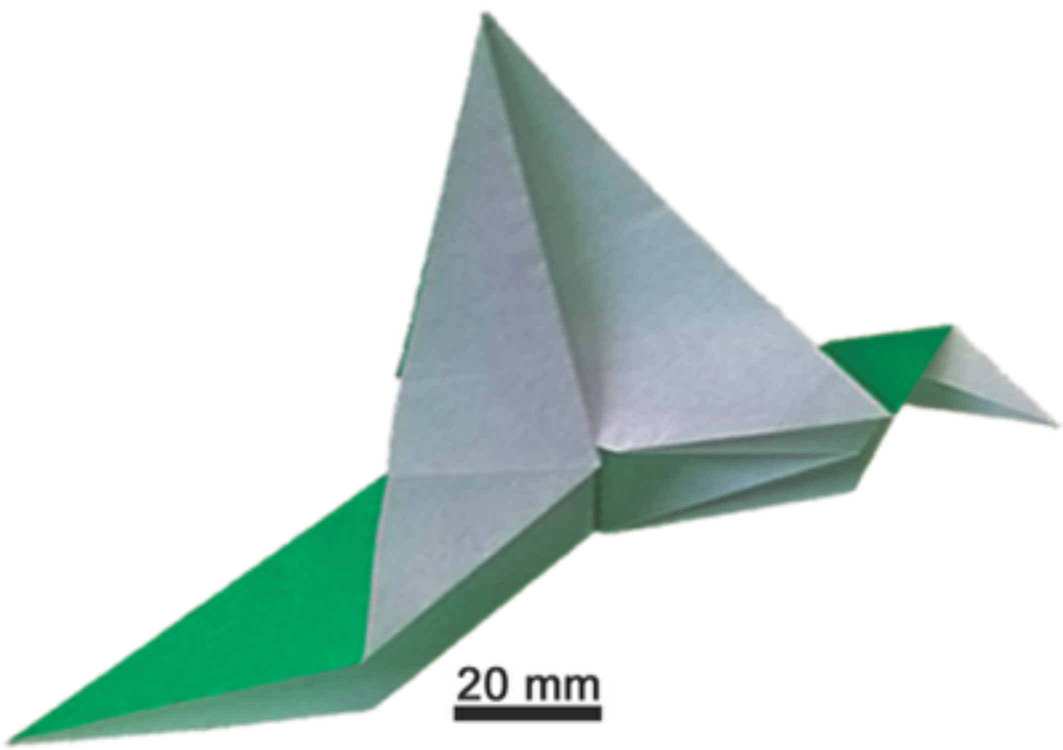

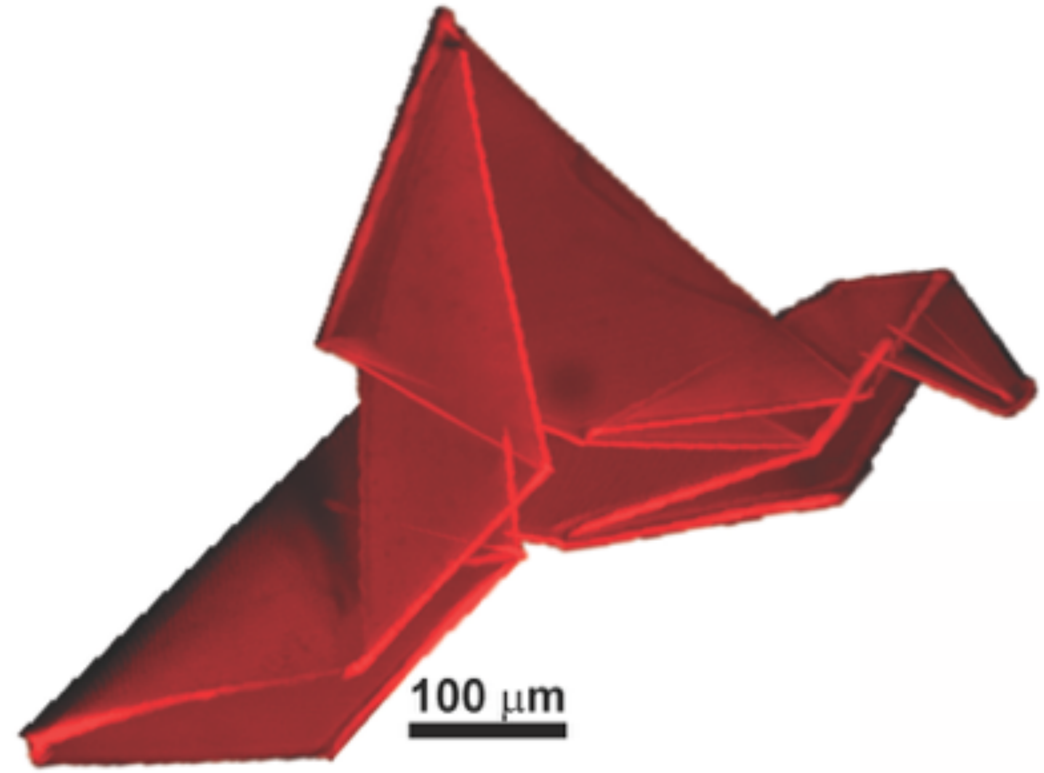

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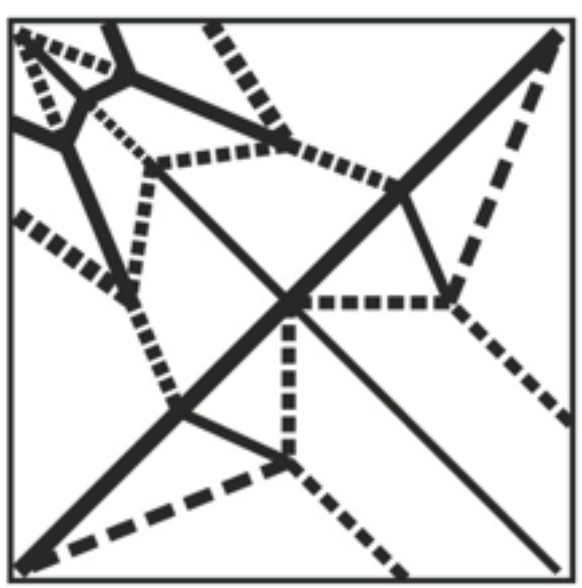
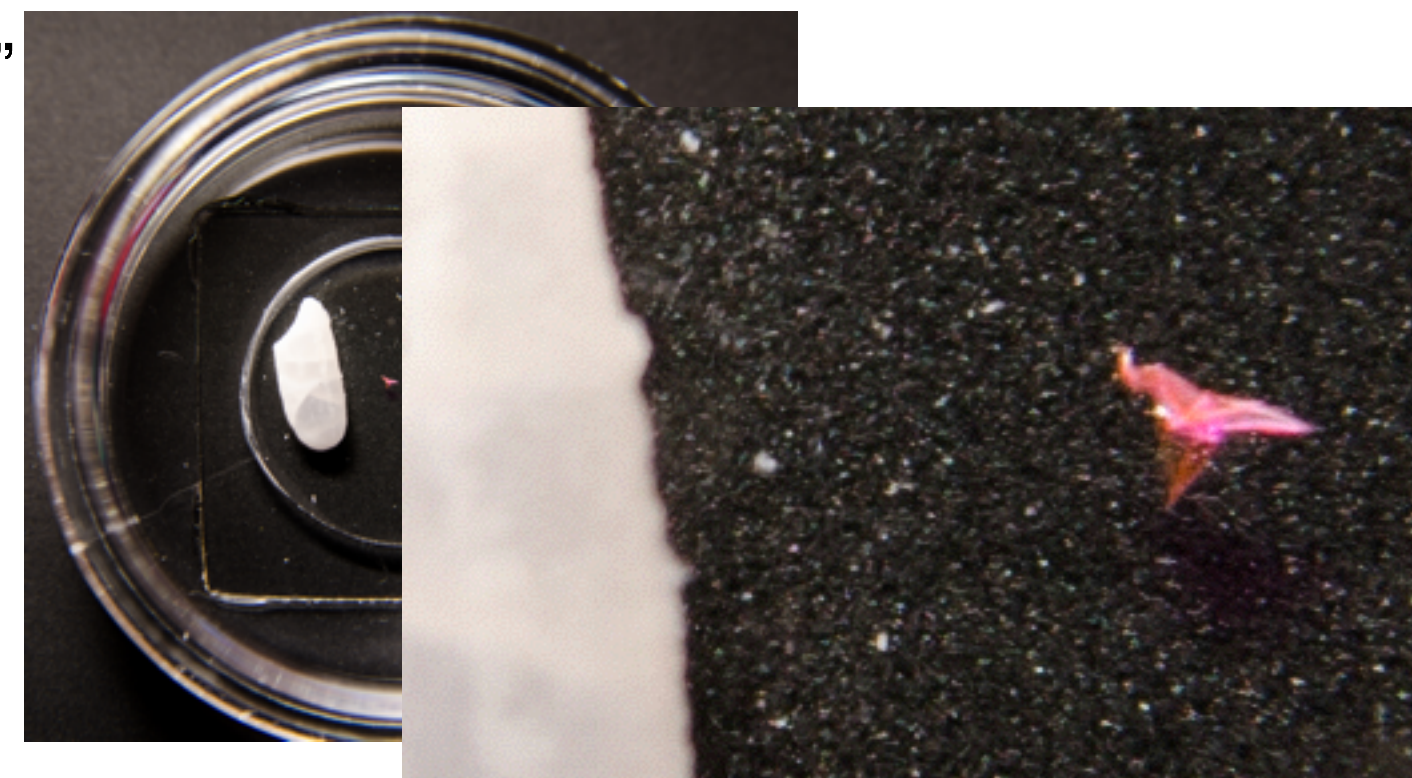
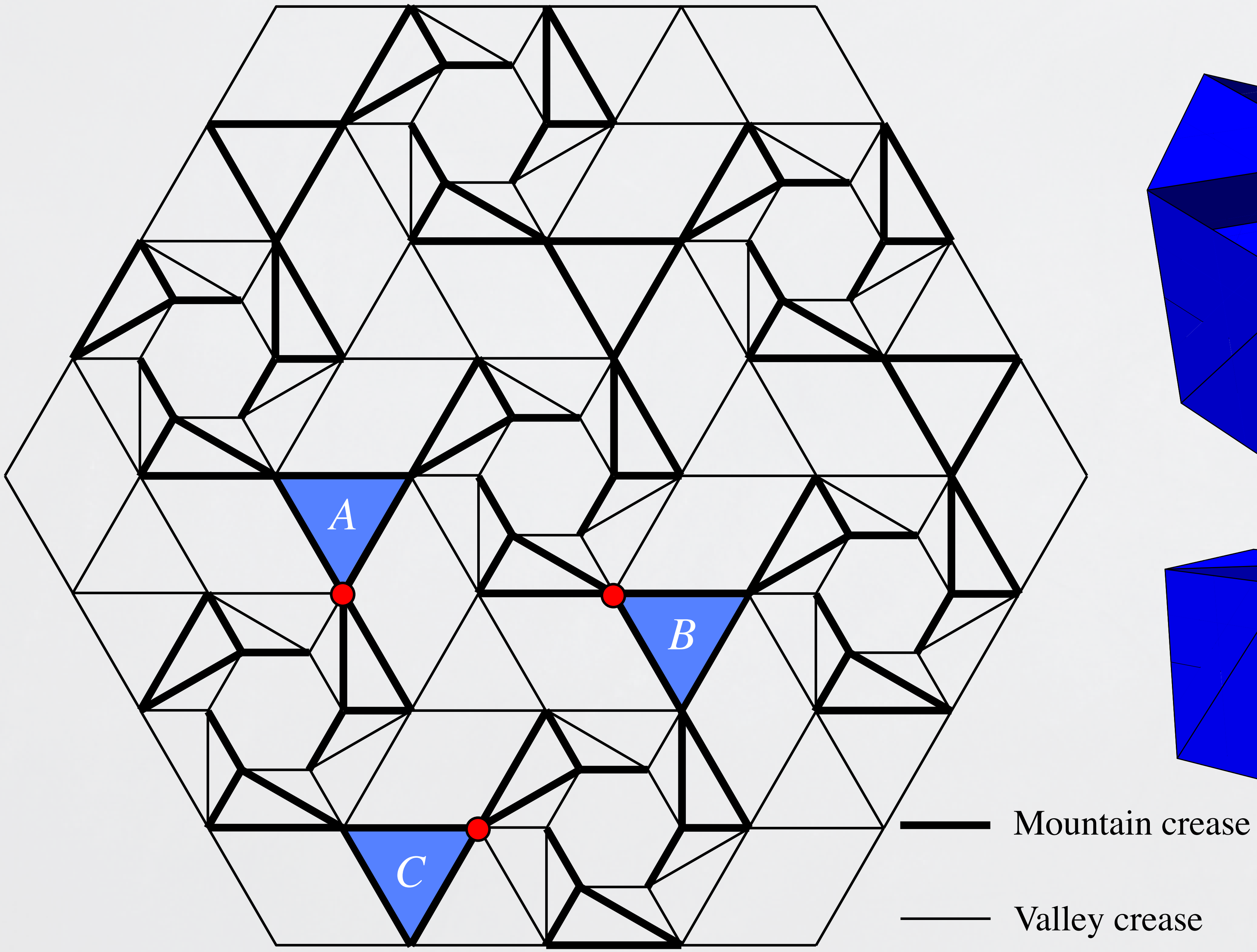
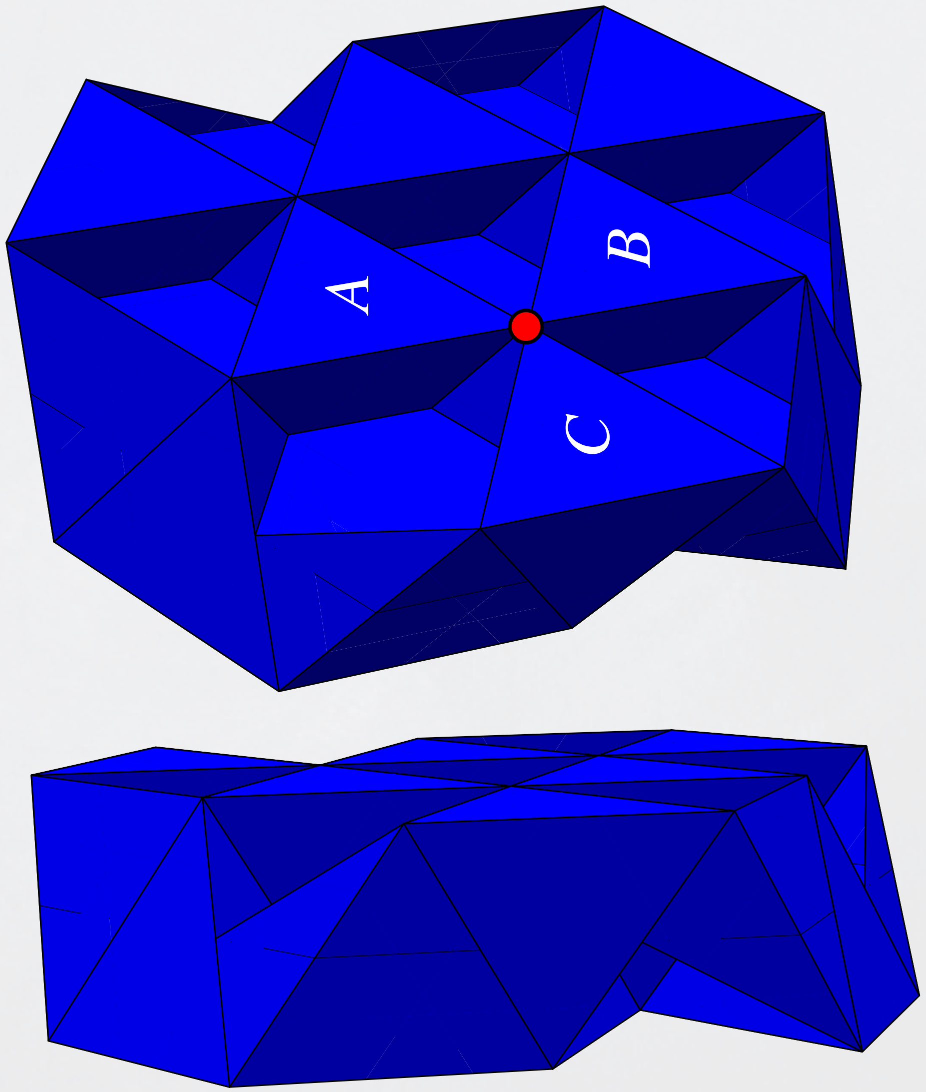


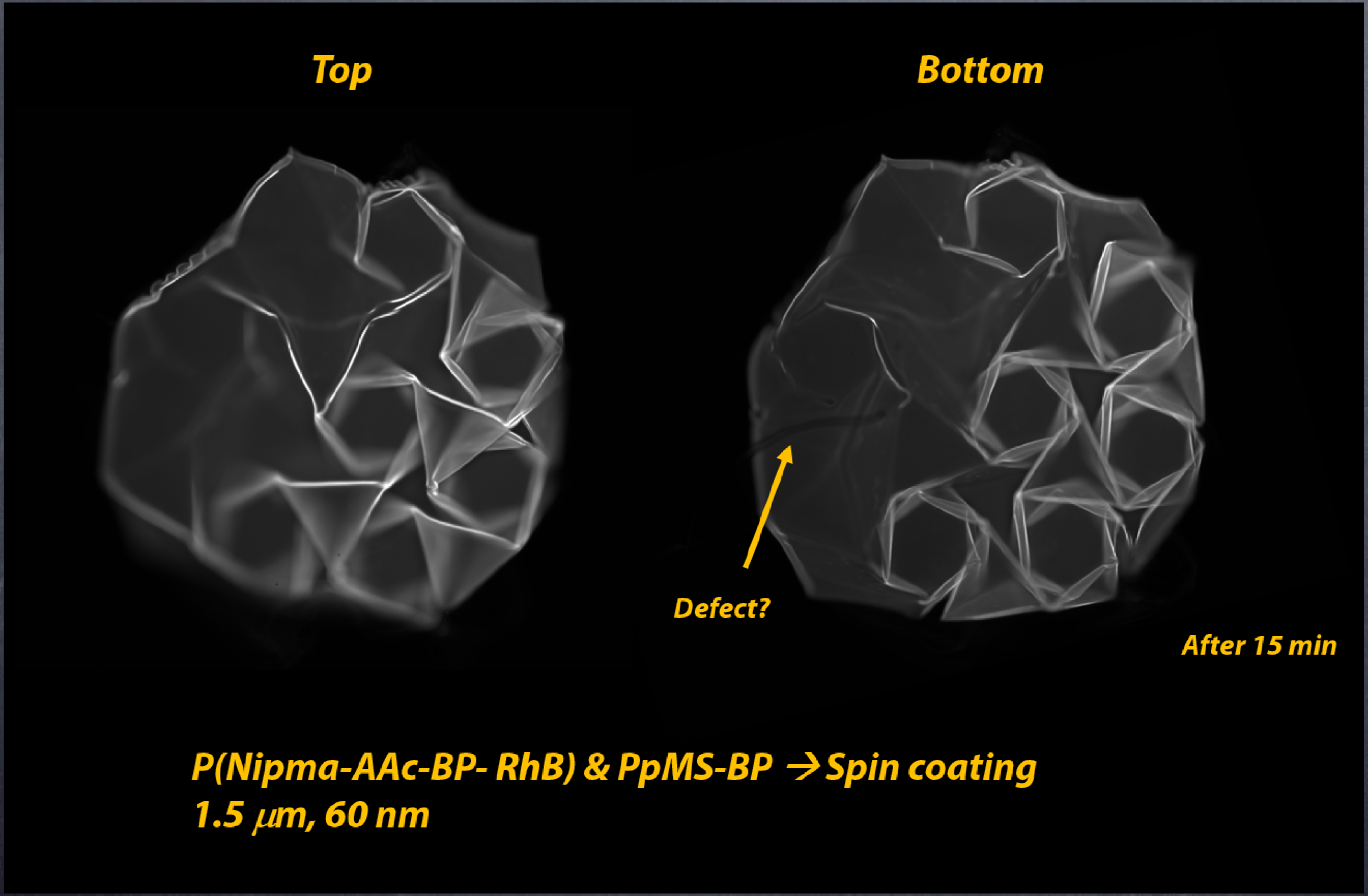


Origami Octahedron-Tetrahedron Truss
(D. Huffman, T. Kawasaki, R. Resch, etc)





Origami Oct-Tet Truss Self-folded in Ryan's Lab (by postdoc Junhee Na)



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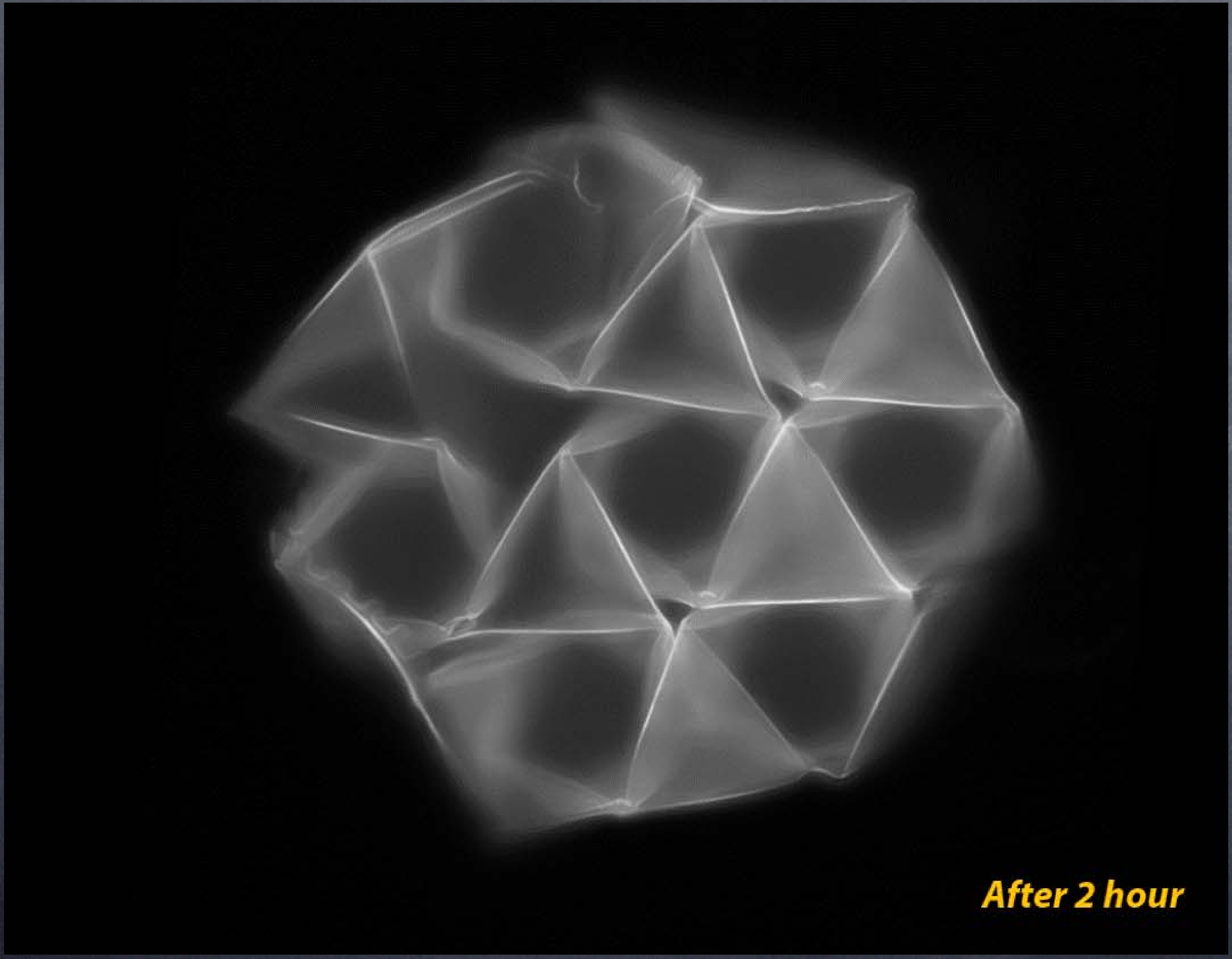
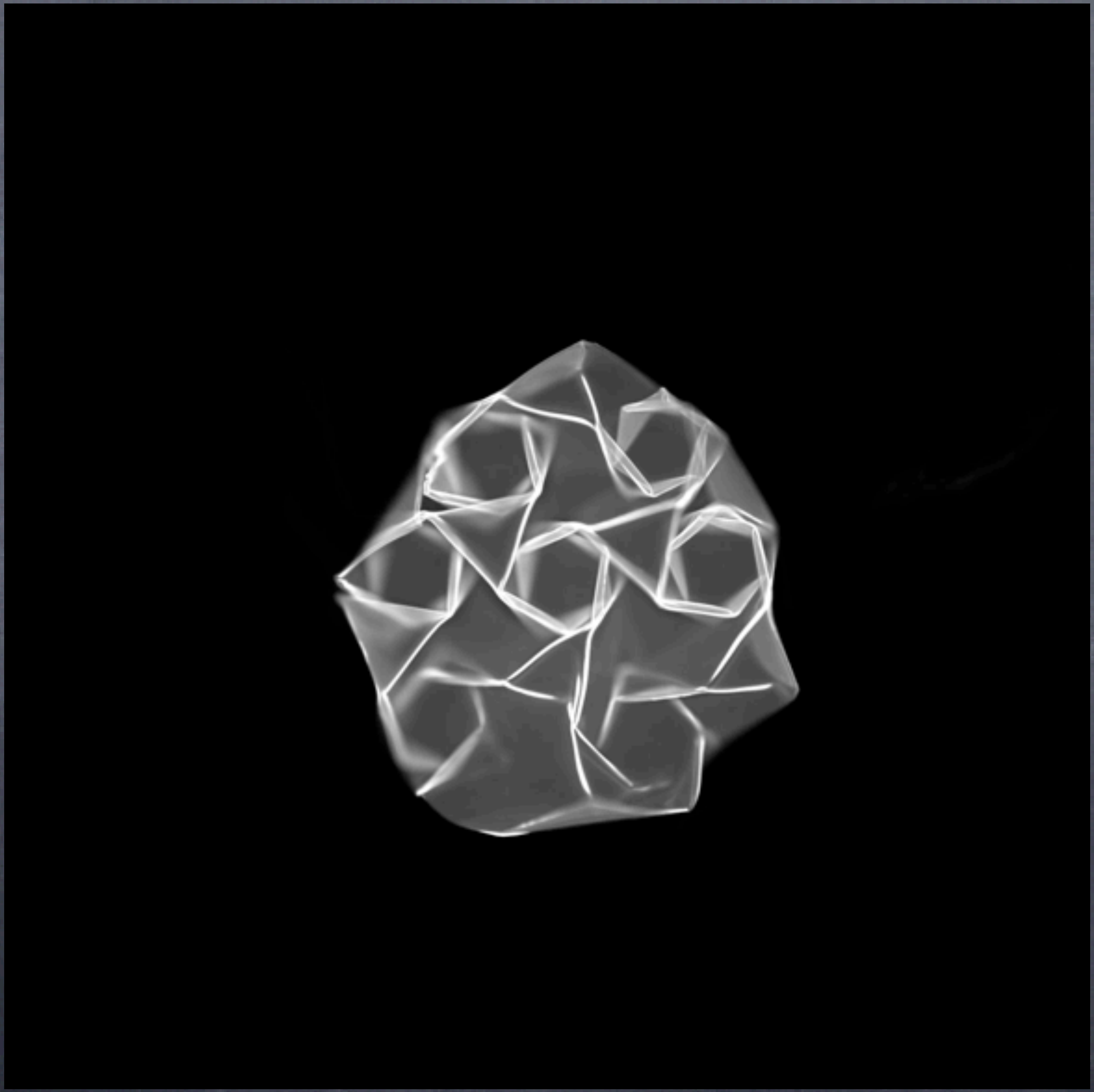
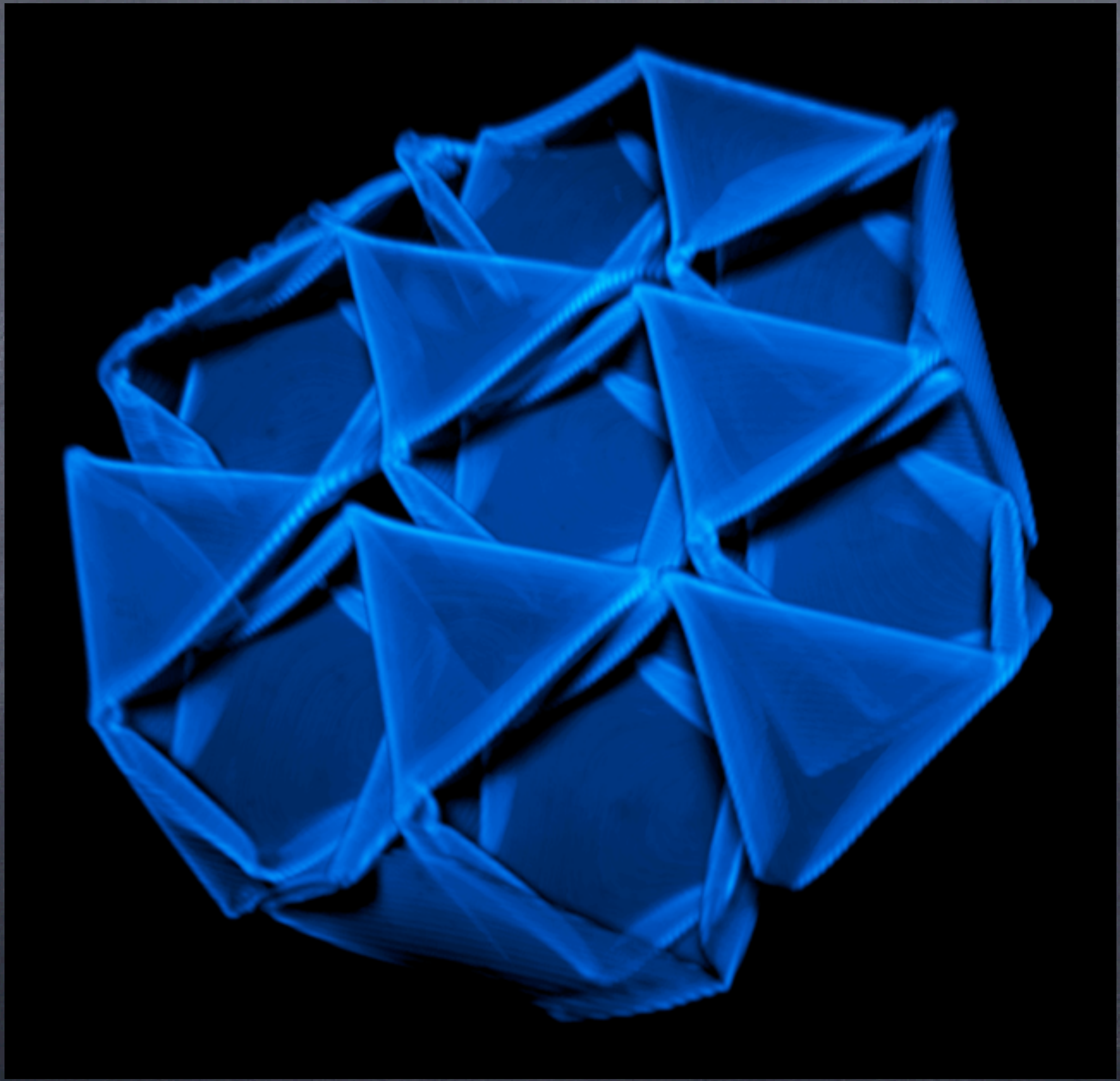


Origami Oct-Tet Truss Self-folded in Ryan's Lab movie of deswelling (Junhee Na)



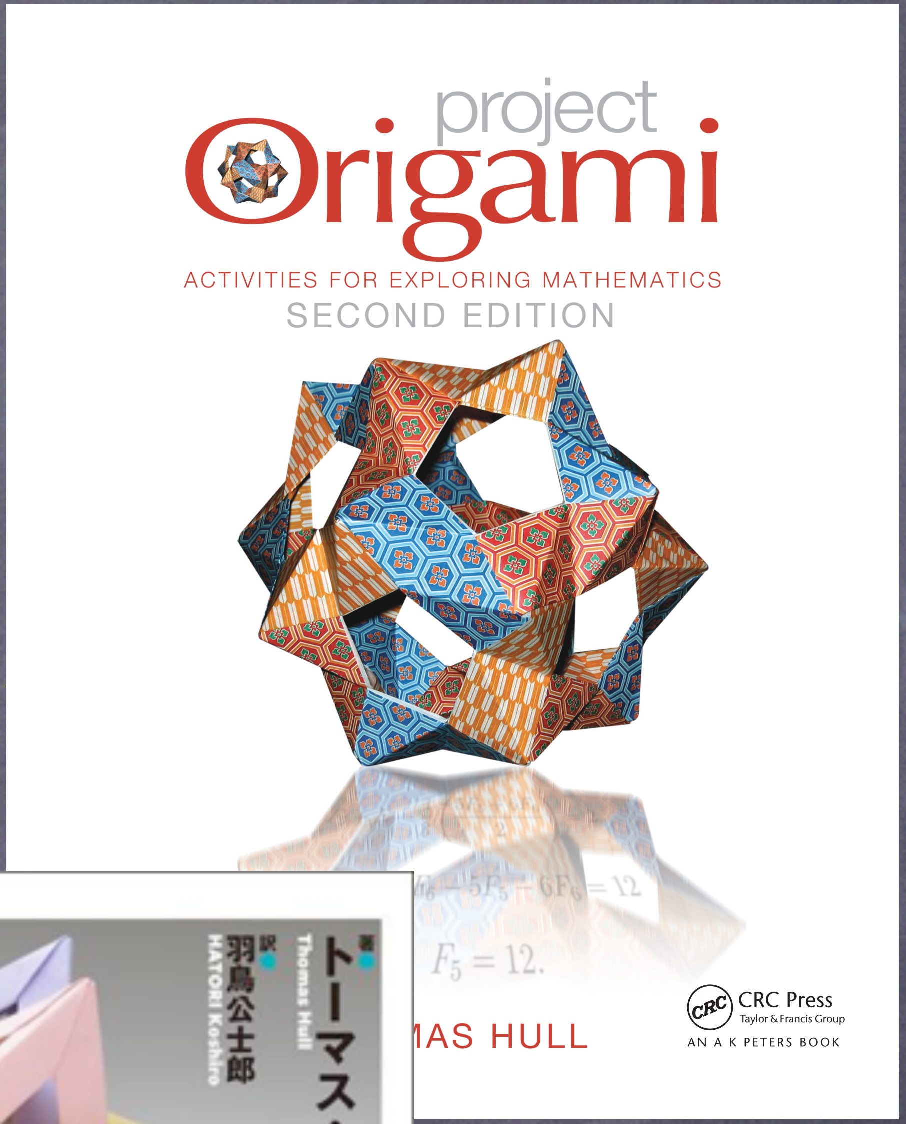
Origami Oct-Tet Truss Self-folded in Ryan's Lab confocal fluorescence microscopy image (Junhee Na)



Thank you!

For more information:

My book, **Project Origami, 2nd ed.**



My web page:
http://mars.wne.edu/~thull

or just email me:
thull@wne.edu